A STABILITY CRITERION OF TIME-DELAY FUZZY SYSTEMS

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ABSTRACT
To guarantee the asymptotic stability, a stability criterion in terms of Lyapunov’s direct method for multiple time-delay fuzzy interconnected systems is proposed in this paper. Each of these systems consists of a number of subsystems represented by Takagi-Sugeno fuzzy models with multiple time delays.

INTRODUCTION
The mathematical models of many physical and engineering systems are frequently of high dimension, or possessing interactive dynamic phenomena. The information processing and requirements to experiment with these models for control purposes are usually excessive. Moreover, the existence of time delays is frequently a source of instability in some way. Hence, the problem of stability analysis of time-delay systems has been one of the main concerns of researchers (see [1-3], for example) wishing to inspect the properties of such systems.

In this paper, we consider a multiple time-delay fuzzy interconnected system composed of J subsystems with interconnections and each subsystem is represented by the so-called Takagi-Sugeno (T-S) fuzzy model with multiple time delays. One critical property of control systems is stability and considerable reports have been issued in the literature on the stability problem of fuzzy dynamic systems (see [4-5] and the references therein). However, a literature survey indicates that the stability problem of fuzzy interconnected systems with multiple time delays has not yet been resolved. Thus, for the purpose of general application, a stability criterion in terms of Lyapunov’s direct method is derived to guarantee the asymptotic stability of multiple time-delay fuzzy interconnected systems.

SYSTEM DESCRIPTION AND STABILITY ANALYSIS
Consider an interconnected system \( F \) composed of \( J \) multiple time-delay subsystems \( F_j, j = 1, 2, \ldots, J \). The \( j \)th subsystem \( F_j \) is described as follows:

\[
\dot{x}_j(t) = f_j(x_j(t)) + \sum_{k=1}^{N_j} g_{kj}(x_j(t - \tau_{kj})) + \sum_{n=1}^{J} \sum_{n \neq j} b_{nj} x_n(t),
\]

where \( f_j \) and \( g_{kj} \) are the nonlinear vector-valued functions; \( x_j(t) \) denotes the state vector and \( x_j^T(t) = [x_{1j}(t), x_{2j}(t), \ldots, x_{nj}(t)] \); \( \tau_{kj} \), the \( k \)th time delay of the \( j \)th subsystem, is a positive real number for \( k = 1, 2, \ldots, N_j \); \( b_{nj} \) is the nonlinear interconnection matrix between the \( n \)th and \( j \)th subsystems.

In a little more than a decade ago, a fuzzy dynamical model had been developed primarily from the pioneering work of Takagi and Sugeno [6] to represent local linear input/output relations of nonlinear systems. Accordingly, the \( j \)th isolated subsystem (without interconnection) of \( N \) is approximated by a fuzzy model described by fuzzy IF-THEN rules. The ith rule of this fuzzy model for the nonlinear interconnected subsystem \( N_j \) is proposed as the following form:

IF \( x_{1j}(t) \) is \( M_{1ij} \) and \( \ldots \) and \( x_{nj}(t) \) is \( M_{nj} \)

THEN \( \dot{x}_j(t) = A_{ij} x_j(t) + \sum_{k=1}^{N_j} A_{ikj} x_j(t - \tau_{kj}) \),

where \( x_j^T(t) = [x_{1j}(t), x_{2j}(t), \ldots, x_{nj}(t)] \), \( i = 1, 2, \ldots, r_j \) and \( r_j \) is the number of IF-THEN rules of the \( j \)th subsystem; \( A_{ij} \) and \( A_{ikj} \) are constant matrices with appropriate dimensions, \( x_j(t) \) is the state vector, \( \tau_{kj} \) denotes the time delay, \( M_{ij}(p = 1, 2, \ldots, g) \) are the fuzzy sets, and \( x_{1j}(t) \)
~ x_j(t) are the premise variables. The final state of this fuzzy dynamic system is inferred as follows:

$$
\dot{x}_j(t) = \sum_{i=1}^{r_j} w_{ij}(t) \left[ A_{ij} x_j(t) + \sum_{k=1}^{N_j} A_{ikj} x_k(t - \tau_{ik}) \right] - \sum_{i=1}^{r_j} w_{ij}(t) \\
= \sum_{i=1}^{r_j} h_i(t) A_{ij} x_j(t) + \sum_{k=1}^{N_j} A_{ikj} x_k(t - \tau_{ik}) \tag{3}
$$

where $w_{ij}(t) = \prod_{p=1}^{r_j} M_{ijp}(x_{ip}(t))$, $h_i(t) = w_{ij}(t) / \sum_{i=1}^{r_j} w_{ij}(t)$, $M_{ijp}(x_{ip}(t))$ is the grade of membership of $x_{ip}(t)$ in $M_{ijp}$. In this paper, it is assumed that $w_{ij}(t) \geq 0$, $i = 1, 2, \ldots, r_j$; $j = 1, 2, \ldots, J$ and $\sum_{i=1}^{r_j} w_{ij}(t)$ for all $t$. Therefore, $h_i(t) \geq 0$ and $\sum_{i=1}^{r_j} h_i(t)$ for all $t$. Therefore, from Eq. (1) and Eq. (3), we have

$$
\dot{x}(t) = \sum_{i=1}^{r_j} h_i(t) A_{ij} x_j(t) + \sum_{k=1}^{N_j} A_{ikj} x_k(t - \tau_{ik}) + \sum_{n=1}^{N_j} b_n x_n(t) \tag{4}
$$

In the following, a stability criterion is proposed to guarantee the asymptotic stability of the multiple time-delay fuzzy interconnected system $F$ which consists of $J$ fuzzy models $F_j (j = 1, 2, \ldots, J)$ described in Eq. (3). Prior to examination of asymptotic stability of $F$, a useful concept is given below.

**Lemma 1** [7]: For any matrices $X$ and $Y$ with appropriate dimensions, we have

$$
X^T Y + Y^T X \leq \kappa X^T X + \kappa^{-1} Y^T Y
$$

where $\kappa$ is a positive constant.

**Theorem 1**: The multiple time-delay fuzzy interconnected system is asymptotically stable, if there exist positive definite matrices $P_j > 0$, $R_{ij} > 0$ and positive constants $\alpha_j > 0$, $\eta > 0$ such that the following inequalities hold:

$$
\lambda_{ij} = \lambda_{ij} (Q_{ij}) < 0; \quad \lambda_{ikj} = \lambda_{ikj} (Q_{ikj}) < 0 \tag{5}
$$

for $i = 1, 2, \ldots, r_j$, $j = 1, 2, \ldots, J$, $k = 1, 2, \ldots, N_j$ where

$$
Q_{ij} = \left( A_j^T P_j \right)^2 + P_j A_j + \sum_{k=1}^{N_j} R_{kj} + \sum_{k=1}^{N_j} \alpha_j P_j A_{ikj} A_{ikj}^T P_j + \eta (J - 1) I + \eta^2 J \left( P_j b_n^T b_n P_j \right), \tag{6}
$$

$$
Q_{ikj} = \alpha_j^{-1} I - R_{kj} \tag{7}
$$

with $P_j = P_j^T$, $R_{ij} = R_{ij}^T$, and $\lambda_{ij} (Q_{ij})$ as well as $\lambda_{ikj} (Q_{ikj})$ denote the maximum eigenvalues of $Q_{ij}$ and $Q_{ikj}$, respectively.

**PROOF OF THEOREM 1**

Let the Lyapunov function for the multiple time-delay fuzzy interconnected system $F$ be defined as

$$
V(t) = \sum_{j=1}^{N_j} v_j(t) = \sum_{j=1}^{N_j} \left[ x_j^T(t) P_j x_j(t) + \sum_{k=1}^{r_j} \int_0^{\tau_{kj}} x_j^T(t - \tau_{kj}) R_{kj} x_j(t - \tau_{kj}) d\tau \right] \tag{8}
$$

where $P_j = P_j^T > 0$ and the weighting matrix $R_{ij} = R_{ij}^T > 0$. We then evaluate the time derivative of $V$ on the trajectories of Eq. (4) to get

$$
\dot{V}(t) = \sum_{j=1}^{N_j} \dot{v}_j(t) = \sum_{j=1}^{N_j} \left[ x_j^T(t) P_j \dot{x}_j(t) + \sum_{k=1}^{r_j} \int_0^{\tau_{kj}} x_j^T(t - \tau_{kj}) R_{kj} \dot{x}_j(t - \tau_{kj}) d\tau \right] \tag{9}
$$

Based on Lemma 1 and Eq. (9), we have

$$
\dot{V} \leq \sum_{j=1}^{N_j} \sum_{i=1}^{r_j} \int_0^{\tau_{kj}} h_i(t) x_j^T(t - \tau_{kj}) A_{ij}^T P_j x_j(t - \tau_{kj}) d\tau + \sum_{j=1}^{N_j} \sum_{n=1}^{N_j} \int_0^{\tau_{kj}} h_i(t) x_j^T(t - \tau_{kj}) R_{nj} x_n(t - \tau_{kj}) d\tau + \sum_{j=1}^{N_j} \sum_{n=1}^{N_j} \int_0^{\tau_{kj}} \eta P_j x_j^T(t - \tau_{kj}) x_n(t - \tau_{kj}) d\tau \tag{10}
$$

Therefore, we have

$$
\dot{V} \leq \sum_{j=1}^{N_j} \sum_{i=1}^{r_j} \int_0^{\tau_{kj}} h_i(t) x_j^T(t - \tau_{kj}) A_{ij}^T P_j x_j(t - \tau_{kj}) d\tau + \sum_{j=1}^{N_j} \sum_{n=1}^{N_j} \int_0^{\tau_{kj}} h_i(t) x_j^T(t - \tau_{kj}) R_{nj} x_n(t - \tau_{kj}) d\tau + \sum_{j=1}^{N_j} \sum_{n=1}^{N_j} \int_0^{\tau_{kj}} \eta P_j x_j^T(t - \tau_{kj}) x_n(t - \tau_{kj}) d\tau \tag{11}
$$

Thus, $\dot{V} \leq 0$, which implies that $V(t)$ is non-increasing. Since $V(t)$ is positive definite and continuous, $V(t)$ converges to a constant value as $t \to \infty$. Therefore, $x_j(t)$ is bounded as $t \to \infty$. This completes the proof of Theorem 1.
\[ + \eta^{-1} x_j^T(t) P_j b_{nj} h_j^T \eta P_j x_j(t) \]  
\[ + \sum_{j=1}^{N_j} \sum_{i=1}^{r_j} \sum_{k=1}^{N_j} h_j(t) x_j^T(t) (A_{ij}^T P_j + P_j A_{ij}) x_j(t - \tau_{kj}) \]  
\[ + \sum_{k=1}^{N_j} \sum_{j=1}^{r_j} \sum_{i=1}^{r_j} \sum_{k=1}^{N_j} h_j(t) x_j^T(t - \tau_{kj}) (\alpha_j^{-1} I - R_{kj}) x_j(t - \tau_{kj}) \]  
\[ + \sum_{j=1}^{N_j} \sum_{i=1}^{r_j} \sum_{k=1}^{N_j} h_j(t) x_j^T(t) [P_j + \eta (J - 1) I] x_j(t) \]  
\[ + \sum_{j=1}^{N_j} h_j(t) x_j^T(t) Q_{ij} x_j \]  
\[ \leq \left( \sum_{j=1}^{N_j} \sum_{i=1}^{r_j} h_j(t) \mathcal{V}_j(t) \right)^2 + \left( \sum_{j=1}^{N_j} h_j(t) \mathcal{V}_j(t) \right)^2. \]  

Based on Eq. (6) and Eq. (7), we have \( \dot{V} < 0 \) and the proof is therefore completed.

**CONCLUSIONS**

This paper is concerned with the stability problem of the multiple time-delay fuzzy interconnected system which consists of a few interconnected subsystems. Each subsystem is represented by a T-S fuzzy models with multiple time delays. A stability criterion in terms of Lyapunov’s direct method is proposed to guarantee the asymptotic stability of multiple time-delay fuzzy interconnected systems.

**REFERENCES**