CRITERIA FOR THE INITIATION OF MODES OF RESPONSE OF A CAISSON SUBJECTED TO A BREAKING WAVE FORCE

Jaw Guei Lin\(^1\) and Chi Chao David Tung\(^2\)

Key words: caisson, breaking wave force, response, initiation criteria.

**ABSTRACT**

This paper concerns problems of a caisson subjected to a breaking wave force: (1) what are the possible modes of response and (2) the conditions initiate into these modes. These two questions have not been studied. The objective of this paper is to answer these two questions. However, the actual response of a caisson that takes place in each mode is not addressed in this study.

A caisson placed on a horizontal frictional base, subjected to a concentrated horizontal force of short duration, applied to the seaward side face of the caisson and accompanied by a corresponding uplift force on the bottom of the caisson, is modeled as a free-standing rigid body. The conditions for the initiation of each mode of motion are derived using the equations of motion of a plane rigid body. The results are given in explicit analytical form and presented graphically. Knowing the magnitude and location of the force, coefficient of friction and the aspect ratio of the body, the mode of motion may be identified easily from the graphs.

**I. INTRODUCTION**

In coastal waters, caissons are used as breakwaters. Under the action of breaking waves, caissons are initiated into various modes of motion. In this paper, the behavior of a caisson is examined in a different way from the works that have hitherto been carried out. Instead of trying to find the actual response of a caisson to a breaking wave force, this study seeks to determine the various possible modes (such as rest, slide, rock and slide-rock) of motion that might take place and, more importantly, the criteria for the initiation of these modes. In an effort to obtain explicit analytical solutions, the models for the caisson and the breaking wave force are made simple. Thus, a caisson is modeled as a rigid body placed on a horizontal frictional base and a breaking wave force is idealized as a horizontal force of short duration acting on the seaward side face of the caisson accompanied by an uplift force on the caisson’s bottom. The criterion for each mode of response is obtained using the three equations of motion of a two-dimensional rigid body.

In a previous publication [2], the same criteria were pursued without including the uplift force. It was shown that the derivations of the criteria were rather involved and the behavior of a caisson was different depending on whether the force is applied above or below the center of mass of the caisson. Thus, the present paper considers only the case of a breaking wave force applied above, but not below, the center of mass of the caisson. Furthermore, only details of the derivation of the criteria for the initiations of the rest, slide and rock modes are presented. The criteria are given analytically and presented graphically.

The paper begins with a section ‘models’ which describes the models employed for the caisson, the breaking wave force and the uplift force. This is followed by sections of derivation of the criteria for the initiation of the three modes. The rest mode (abbreviated \(RE\)) is given in Section III, the slide mode (abbreviated \(SL\)) in Section IV and the Rock about point \(O\) mode (abbreviated \(RO\)) in Section V. These sections are given to demonstrate what is involved in the derivations. While the derivations for the rest and slide cases are simple, that for the rock about point \(O\) case is lengthy and complex. In addition to these three modes of response, there are other modes, namely, the slide-rock about point \(O\) mode (\(SRO\)), the rock about point \(O\) mode (\(RO\)) and the slide-rock about point \(O\) mode (\(SRO\)). The derivations for these modes are understandably complex and lengthy, and are not included in this paper. Those derivations can be found in a report [6]. The report also contains results when all modes (including the ones not covered in this paper) are combined.
II. MODELS

Referring to Fig. 1, consider a plane rigid body, partially immersed in water, of uniform mass distribution, the total mass in water being \( m \). The body is rectangular in elevation and footprint whose height is \( 2H \), width \( 2B \) and depth is equal to unity. It rests on a horizontal frictional base. The friction between the base and the body is of the Coulomb type with coefficient of static friction \( \mu \). The body is initially at rest and is subjected to an impact pressure from breaking wave on the seaward side of the vertical wall. The resultant of the pressures is the horizontal force \( F \) which is assumed to act on the body only for a short duration. The force, for convenience, is expressed in terms of the weight of the body and its rotation,

\[
\text{Horizontal and vertical displacements of the center of mass}
\]

The motion of the plane body is specified by the horizontal and vertical displacements of the center of mass \( \xi \) and \( \eta \), considered positive in the counterclockwise direction from positive x-axis. The horizontal and vertical displacements of \( C \) are \( x \) and \( y \), considered positive to the right and upwards, respectively as shown in Fig. 1. The reaction forces are \( f_x \) and \( f_y \), positive to the right and upwards, respectively. \( f_x \) acts at a distance \( \xi \) from \( C \). The uplift force \( U \) acts at a distance \( B/3 \) from \( C \).

and, by taking moment of the forces about \( C \),

\[
f_x H + f_y \xi + F h + UB/3 = 0 \tag{3}
\]

By substituting Eqs. (1) and (2) into Eq. (3), we get

\[
\xi = -\frac{Bk[(\frac{\gamma(1+k')}{4} + q/3]}{1-qk}
\tag{4}
\]

where \( \gamma = H/B \) is the aspect ratio of the body. Several conditions for the body to be at rest mode must be satisfied. These are:

1. The body should be in contact with the base. That is, \( f_y \) must be greater than or equal to zero.
2. The body should not be sliding. That is, \( f_x \) must be smaller than or equal to the limiting Coulomb friction force \( \mu f_y \).
3. The resultant vertical force should remain within the base \( (OO') \).

For condition 1, \( f_y \geq 0 \), and from Eq. (2), we get

\[
k \leq \frac{1}{q} \tag{5}
\]

For condition 2, \( f_x \leq \mu f_y \), we get

\[
\mu \geq \frac{f_x}{f_y} = \frac{k}{1-qk} = \mu_A(k) \tag{6}
\]

Finally, for condition 3, \( f_y \) must lie within the base \( (OO') \) of the body. That is, \( |\xi| \leq B \). From Eq. (4), the condition requires

\[
k \leq \frac{1}{\gamma(1+k') + 4q/3} = k_A \tag{7}
\]

It may be verified that \( k_A \leq 1/q \).

The above conditions of Eqs. (6) and (7), constitute the criteria for the body to remain at rest under the action of \( F \) and \( U \). These conditions may be conveniently presented graphically as a region using the parameters \( k \) and \( \mu \) as the horizontal and the vertical axes respectively as shown in Fig. 2.

In Fig. 2, the curve \( OA \) (or \( \mu_A(k) \)) and the line \( AH \) (or \( k = k_A \)) intersect at point \( A \) with coordinates \( k = k_A \) and \( \mu = \mu_A \).

\[
\mu = \mu_A = \frac{1}{\gamma(1+k') + 4q/3} \geq k_A \tag{8}
\]

The region that represents the rest mode is shaded and denoted by the symbol \( RE \) in Fig. 2.

From Eqs. (7) and (8) and Fig. 2, we can see that the larger
the values of $\gamma$, $k'$ and $q$, the closer is the line $AH$ to the $\mu$ axis, the narrower is the rest region and the less likely is the body to remain at rest. Also, when $q = 0$, the result agrees with those obtained earlier in [5] and [6].

IV. SLIDE MODE (SL)

The equations of motion for the initiation of a slide mode are the same as those for a rest mode except the equation in the $x$-direction. They are:

$$m\ddot{x} = f_x - F$$  \hspace{1cm} (9)
$$f_y = mg - U = mg(1-qk)$$  \hspace{1cm} (2)

and,

$$f_xH + f_y\xi + Fh + UB/3 = 0$$  \hspace{1cm} (3)

Here and hereafter, over-dot denotes differentiation with respect to time.

Three conditions for a slide mode to occur are

1. The body should be in contact with the base, that is, $f_y \geq 0$.
2. Horizontal force should overcome the limiting Coulomb friction force $\mu f_y$ when the body starts to slide. That is, $f_x = \mu f_y$.
3. The resultant vertical force should remain within the base $(OO')$, that is, $|\xi| \leq B$.

The condition $f_y \geq 0$ gives, from Eq. (2), $k \leq 1/q$. Eq. (3) gives

$$\xi = -k(k'\gamma + qB/3) + \mu H(1-qk)$$  \hspace{1cm} (10)

For $k \leq 1/q$, $\xi$ should always be smaller than or equal to zero, that is, $\xi \leq 0$. The condition $|\xi| \leq B$ therefore requires,

$$\mu \leq \frac{(1/\gamma) - k(k' + (4q/3)\gamma)}{1-qk} = \mu_i(k)$$  \hspace{1cm} (11)

The curve $\mu_i(k)$ is sketched in Fig. 3 in the $k - \mu$ plane; it intersects $\mu_0(k)$ at point $A$, and the horizontal $k$ axis at $D$ where the abscissa is

$$k_D = \frac{1}{\gamma k' + 4q/3}$$  \hspace{1cm} (12)

Beyond $k_D$, the curve $\mu_i$ is negative and goes to negative infinity as $k$ approaches $1/q$.

The region corresponding to a slide mode is the shaded area $OAD$ in Fig. 3. The symbol $SL$ is used to denote the slide mode. In region $OAM$, the rest mode governs because the horizontal reaction force $f_x$ in a rest mode is smaller than the horizontal reaction force $f_x$ in the slide mode. From Eq. (9), we have $\ddot{x} = (1-qk)(\mu - \mu_0)$. In the region of slide mode, ($OAD$), $k \leq 1/q$ and $\mu \leq \mu_0$; thus, $\ddot{x} \leq 0$. Since the body is originally at rest, $\ddot{x} \leq 0$ and $x \leq 0$. That is, the body slides to the left under the action of $F$, as expected.

V. ROCK MODE ABOUT POINT $O$ (RO)

When a body is about to rock about point $O$, the equations of motion are:

$$m\ddot{x} = f_x - F$$  \hspace{1cm} (9)
$$m\ddot{y} = f_y - mg + U$$  \hspace{1cm} (13)

and, noting that $f_x$ acts at point $O$ (see Fig. 1) about which the body rotates,

$$I\ddot{\theta} = f_xH - f_yB + Fh + UB/3$$  \hspace{1cm} (14)

Here, $I = m(B^2 + H^2)/3$ is mass moment of inertia of the body about its center of mass $C$.

The accelerations $\ddot{x}$ and $\ddot{y}$ of point $C$ are related to the angular acceleration $\ddot{\theta}$ of the body as $\ddot{x} = -H\ddot{\theta}$ and $\ddot{y} = B\ddot{\theta}$. Eq. (14) gives

$$\ddot{\theta} = \frac{3g}{4B(1+\gamma)} \left( \frac{k}{k_A} - 1 \right)$$  \hspace{1cm} (15)
where \( k_A \) is given in Eq. (7).

Eqs. (9) and (10) give respectively

\[
f_x = \frac{mg}{4(1 + \gamma^2)}(ak + b) \tag{16}
\]

and

\[
f_y = \frac{mg}{4(1 + \gamma^2)}(ck + d) \tag{17}
\]

where

\[
a = 4 + \gamma^2 - 3\gamma k' - 4\gamma q \tag{18}
\]

\[
b = 3\gamma \tag{19}
\]

\[
c = 3\gamma(1 + k') - 4\gamma^2 q \tag{20}
\]

and

\[
d = 1 + 4\gamma^2 \tag{21}
\]

It is noted that both \( f_x \) and \( f_y \) are either positive or negative since the quantities \( a \) and \( c \) may be positive or negative. For the case of \( c \geq 0, f_x \) is always greater than or equal to zero. For the case of \( c \leq 0, f_x = -\gamma k + d \), in which case, \( f_x \geq 0 \) for \( k \leq d/\gamma \). For \( k = d/\gamma \), the body is in a free-flight mode.

For the body to rock about point \( O \), \( \dot{\theta} \) must be greater than or equal to zero. This means, from Eq. (15),

\[
k \geq k_A \tag{22}
\]

For a rock mode to be initiated, \( f_x \) must not exceed the limiting friction force. That is, \( |f_x| \leq \mu f_y \), or,

\[
\mu \geq \frac{|ak + b|}{ck + d} = \mu^*(k) \tag{23}
\]

The function \( \mu^*(k) \) behaves in a variety of ways depending on the signs of the quantities \( a, c \) and \( f_x \). There are altogether six cases that must be considered. They are:

Case I: \( a \geq 0, c \geq 0, f_x \geq 0, \mu^* = (ak + b)/(ck + d), 0 \leq k \leq \infty \)

Case II: \( a \geq 0, c \leq 0, f_x \geq 0, \mu^* = (ak + b)/(-ck + d), 0 \leq k \leq d/\gamma \); for \( d/\gamma \leq k \leq \infty \), the body is in a free-flight mode

Case III: \( a \leq 0, c \geq 0, f_x \geq 0, \mu^* = (-a|k + b|/(ck + d), 0 \leq k \leq b/\gamma) \)

Case IV: \( a \leq 0, c \leq 0, f_x \geq 0, \mu^* = (-a|k + b|/(-ck + d), 0 \leq k \leq b/\gamma) \)

Case V: \( a \leq 0, c \geq 0, f_x \leq 0, \mu^* = -a|k + b|/(ck + d), b/\gamma \leq k \leq \infty \)

Case VI: \( a \leq 0, c \leq 0, f_x \leq 0, \mu^* = -a|k + b|/(-ck + d), b/\gamma \leq k \leq \infty \), for \( d/\gamma \leq k \leq \infty \), the body is in a free-flight mode

Properties of \( \mu^*(k) \) are examined for each of the above six cases.

**Case I:**

The curve \( \mu^*(k) = (ak + b)/(ck + d) \) passes point \( A \) (see Fig. 2). As \( k \) approaches infinity, \( \mu^* = (a/c) \geq 0 \). The slope of \( \mu^*(k) \) is \( d\mu^*/dk = (ad - bc)/(ck + d)^2 \) where \( ad - bc = 4(1 + \gamma^2)e \) is independent of \( k \); here

\[
e = 1 + \gamma^2 - 3\gamma k' - \gamma q \tag{24}
\]

which may be greater or smaller than zero. Thus the slope of \( \mu^*(k) \) is a decreasing function of \( k \) and approaches zero as \( k \) approaches infinity. Also, the slope of \( \mu^*(k) \) at point \( A \) is equal to \( e/4(1 + \gamma^2) \). Since the quantity \( e \) may be greater or smaller than zero, distinction must be made between two sub-cases: sub-case 1 is for \( e \geq 0 \), denoted by case I1 and sub-case 2 is for \( e \leq 0 \) denoted by case I2. Figs. 4(a) and 4(b)
show, from Eqs. (22) and (23), the regions for rock mode to occur for cases II and I2, respectively.

For values of $\gamma$, $k'$ and $q$, if $k$ (force) and $\mu$ (coefficient of friction) correspond to a point in the $k - \mu$ plane that falls in the region to the right of the vertical line $AH$ and above the curve $\mu = \mu^*(k)$ in Figs. 4(a) and 4(b), a rock mode would ensue. Here the region corresponding to a rock (about point $O$) mode is shaded and the symbol $RO$ is used to denote the case of rock (about point $O$) mode.

**Case II:**

In this case, $\mu^* = (ak + b)(-c |k + d|)$ passes the point $A$ and the expression of its slope is the same as that in case I. For $k \leq d/|c|$, $\mu^*$ is greater than or equal to zero and approaches infinity at $k = d/|c|$. Thus, the slope of $\mu^*$ is greater than or equal to zero and consequently the quantity $e$ is either greater than or equal to zero. It may be verified that $d/|c| \geq 1/q$. The region corresponding to a rock mode is identified in Fig. 5 as that to the right of $AH$ above $\mu^*$ and to the left of the line $PQ$ ($k \leq d/|c|$), shaded and denoted by $RO$. For $k \geq d/|c|$, the body would be lifted off the base in a free-flight mode denoted by the symbol $FF$.

**Case III:**

In this case, $\mu^* = (\alpha k + b)/(-c |k + d|)$ for $k \leq d/|c|$ and $\mu^* = b/|a|$. Again, $\mu^*(k)$ passes point $A$ and is equal to zero at $k = b/|a|$; the expression of its slope is the same as that in case I and hence $e \leq 0$. It may be verified that $(b/|a| - 1/q) = 0$ and $\mu^* = 0$. The shaded regions shown in Figs. 7(a), 7(b) and 7(c) correspond respectively to a rock mode for these cases.

**Case IV:**

In this case, $\mu^* = (\alpha k + b)/(-c |k + d|)$ for $k \leq d/|c|$ and $\mu^* = b/|a|$. By comparing $d/|c|$, $b/|a|$ and $1/q$, we see that $d/|c| - b/|a| = \mu^* - b/|a| = 4(1 + \gamma^2)/\alpha c$, $\mu^* = 0$ and $d/|c| - b/|a| = [q + 3\gamma(1 + k')]/|k|q$ which is always greater than zero. Thus, case IV has three sub-cases: sub-case IV1 is for $e \geq 0$, $e' \geq 0$ in which case $b/|a| \geq d/|c|$ and $b/|a| \geq 1/q$; sub-case IV2 is for $e \leq 0$, $e' \geq 0$ in which case $b/|a| \leq d/|c|$ and $b/|a| \geq 1/q$; finally, sub-case IV3 is for $e \leq 0$, $e' \leq 0$ in which case $b/|a| \leq d/|c|$ and $b/|a| \leq 1/q$.

In case IV1, the characteristics of $\mu^*$ are the same as those in case II. In cases IV2 and IV3, $\mu^*$ passes point $A$; for these cases, $e \leq 0$, the slope of $\mu^*$ decreases as $k$ increases.

Also, at $k = b/|a|$, $\mu^* = 0$.

The shaded regions shown in Figs. 7(a), 7(b) and 7(c) correspond respectively to a rock mode for these cases.

**Case V:**

In this case, $a \leq 0$, $c \geq 0$, $f \leq 0$, $\mu^* = -\alpha k + b)/(\alpha c - d)$ and $b/|a| \leq k \leq \infty$. The characteristics of $\mu^*$ are: at $k = b/|a|$, $\mu^* = 0$; as $k$ approaches infinity, $\mu^* = 0$ and its slope approaches zero. Since $\mu^* \geq 0$, its slope must be greater than or equal to zero as well. Thus the quantity $e$ must be less than
Since $\frac{b}{|a|} \leq \frac{1}{q}$ and $k \leq d / |c|$, it is seen that at $k = \frac{b}{|a|}$, $\mu^* = 0$, and $\mu^*$ approaches infinity at $k = d / |c|$. Since the slope of $\mu^*$ is equal to $d \mu^* / dk = -4(1 + \gamma^2) e^{(-\frac{1}{2}k + d)^2}$ and must be greater than or equal to zero, we conclude that $e \leq 0$ indicating that $b / |a| \leq d / |c|$. However, since $b / |a|$ may be greater or smaller than $1 / q$, two sub-cases arise. The characteristics of $\mu^*$ are such that $\mu^*$ starts at zero at $k = \frac{b}{|a|}$ and slopes up and approaches infinity at $k = d / |c|$ as shown in Figs. 9(a) and 9(b) for $b / |a| \geq 1 / q$ and $b / |a| \leq 1 / q$, respectively. In both Figs. 9(a) and 9(b), the shaded regions to the right of the line $NL$ and above $\mu^*$ correspond to a rock mode. For $k \geq d / |c|$ the body is in a free-flight mode.

The six cases are divided based on the sign of $a$ and $c$ and that of $f_x$. In the analysis, it is seen that the sub-cases also depend on the sign of the quantities $e$ and $e'$. Since these quantities are all functions of $\gamma$, $k'$, and $q$, we identify these cases in a $k'–q$ plane with $\gamma$ as the parameter as shown in Fig. 10. The lines $a = 0$, $c = 0$, and $e = 0$, and $e' = 0$ are indicated.

VI. DISCUSSIONS AND CONCLUDING REMARKS

1. If we let $q = 0$, the results of the present paper which
includes uplift force, reduce to those in [2] and [5] where uplift force is not considered.

2. Since the caisson being considered is rectangular in its elevation and of uniform mass distribution, $k'$ can not exceed unity.

3. The present study of the behavior of a caisson considering uplift force should be extended to cover the case of force $F$ applied below the center of mass of the caisson.

4. The results of this study contribute to a better understanding of the behavior of a caisson under the action of an impact force. The study is not concerned with the problem of determining the actual response of the caisson. For example, the study does not address the issue of the amount of sliding or rocking a caisson would undergo. The study, however, provides information based on which the engineer can make design decisions. For example, the configurations of the caisson may be adjusted to avoid it being initiated into a mode of response that is not considered desirable. More significantly, given the magnitude and location of the force $F$, the uplift force, and the coefficient of friction between the caisson and the base, one may, based on the region of rest (in the $k - \mu$ plane), determine the dimensions of the caisson (See [3] and [4]).

ACKNOWLEDGMENTS

The study is suggested and supported in part by the Harbor and Marine Technology Center, Institute of Transportation under the project entitled ‘A Study of Harbor Resonance of Hwa-Lien Harbor’.

REFERENCES