Journal of Marine Science and Technology, Vol. 25, No. 4, pp. 424-434 (2017)
DOI: 10.6119/JMST-017-0324-1

MIXED H_2/*PASSIVITY* PERFORMANCES CONTROL OF DISCRETE-TIME LINEAR STOCHASTIC SYSTEMS

Cheung-Chieh Ku and Rui-Wen Chen

Key words: mixed performance control, multiplicative noise term, passivity theory, H_2 scheme.

ABSTRACT

A mixed performance control problem of discrete-time linear stochastic systems is discussed and investigated subject to H_2 and passivity performances in this paper. Based on Itô modeling approach, stochastic systems can be represented as deterministic difference equation with multiplicative noise term. For the stochastic systems, H_2 minimization problem and passivity constraint are simultaneously considered to achieve minimum output energy and attenuation performance. Applying Lyapunov theory, some sufficient conditions are derived into extended Linear Matrix Inequality (LMI) form to apply convex optimization algorithm. Moreover, a mixed H_2/*Passivity* performance controller can be designed such that asymptotical stability and required performances of closed-loop system are guaranteed in the mean square. Finally, some simulations are proposed to demonstrate effectiveness and applicability of the proposed design method.

I. INTRODUCTION

System stability and control performance are two important issues in control engineering. For stability issue, Lyapunov function provides a powerful tool for linear systems and nonlinear systems. On the other hand, various schemes have been employed to achieve the required performance. For example, H_2 scheme (Peres and Geromel, 1993; Du, 2006; Ma and Chen, 2006) is applied to minimize a quadratic control performance index. The H_∞ scheme (Mahmoud, 2000; Li and Ugrinovskii, 2007; Willmann et al., 2007) proposes performance index to achieve robustness of systems. Moreover, some mixed H_2/H_∞ performance schemes were developed by Kim (2001), Yang et al. (2002), Qiu (2008), Fioravanti et al. (2014) and Orihuela et al. (2015) through combining the merits of optimal H_2 and robust H_∞ control schemes. Usually, the purpose of mixed H_2/H_∞ performance scheme is to minimize upper bound of H_2 performance under a desired H_∞ norm bound constraint. Thus, the mixed H_2/H_∞ performance scheme is a more attractive design method in engineering practice because sole control performance is a worst case design that leads conservative. In order to extend the application of mixed performance controller design method, its generality and flexibility are an interesting issue.

To propose a general and flexible mixed performance control criterion, the passivity theory (Jiang and Hill, 1998; Xie et al., 1998; Lozano et al., 2000; Tan et al., 2010) is considered for achieving optimal performance in this paper. Referring to (Lozano et al., 2000), Willems developed power supply function expressing passivity theory based on conservation, dissipation and transport of system energy. In general, passivity theory provides useful tool to analyze stability of linear systems and nonlinear systems. Based on the energy concept, the passivity theory was furtherly applied to deal with attenuation performance for fuzzy control (Li et al., 2005; Ku et al., 2010), observer-based control (Mathiyalagan et al., 2015), filter design (Wang et al., 2016) and so on. Through setting power supply function (Lozano et al., 2000; Ku et al., 2010), it is easily found that the passivity theory provides a formulation including H_∞ constraint, positive real theory and passive types. It should be noted that the H_∞ scheme is a special case of passivity theory. According to the above description, the passivity theory is applied to constrain the effect of external disturbance on the system in this paper. Thus, a mixed H_2/*Passivity* performance control criterion is proposed to ensure H_2 performance under the desired disturbance attenuation performance. The similar mixed H_2/*Passivity* performance controller design method was developed by Ku and Li (2015) and Ku (2016) for continuous-time deterministic systems. To extend the applicability of Ku and Li (2015) and Ku (2016), mixed performance control problem of the discrete-time linear stochastic system is discussed and solved in this paper.

In practical control, stability analysis and controller synthesis of stochastic systems are always challenging problems according to characteristics as unmeasurable and unpredictable dynamics. Since the development of Itô stochastic modelling approach

Paper submitted 04/22/16; revised 02/13/17; accepted 03/24/17. Author for correspondence: Cheung-Chieh Ku (e-mail: ccku@mail.ntou.edu.tw).
Department of Marine Engineering, National Taiwan Ocean University, Keelung County, Taiwan, R.O.C.

(Chung and Chang, 1990; Karatzas and Shreve, 1991; Ghaoui, 1995; Lu and Skelton, 2002; Xu et al., 2004; Liu et al., 2008), a term multiplying noise and state in stochastic difference equation (Karatzas and Shreve, 1991) is proposed to describe stochastic behaviors. Because the multiplicative noise is more practical and realistic than traditional additive noise, stochastic difference equation was widely applied to describe stochastic systems such as communication systems, aerospace systems and image processing systems. Many control fundamentals of deterministic systems were extended to analyze the stability of stochastic systems (Chung and Chang, 1990; Ghaoui, 1995; Xu et al., 2004; Liu et al., 2008; Wang and Zhu 2015; Zhu and Cao, 2015). Based on the description of stochastic system, the stability of Markovian jump neural networks has been discussed (Zhu and Cao 2012 a, b; Zhu et al., 2017). However, some extra limits are required to deal with stochastic processes. One of the extra limits is the sense of mean square (Lu and Skelton, 2002) which is applied to define stability concept of stochastic systems. To the best of our knowledge, the mixed H_2/*Passivity* performance control problem of discrete stochastic systems has not been solved in the literature. Moreover, it is an interesting and worth control issue to be discussed and investigated.

Motivated by the above illustration, a mixed H_2/*Passivity* performance controller design method is proposed for the discrete-time linear stochastic systems. Based on the proposed design method, the minimum H_2 performance is achieved under the desired passivity constraint in the mean square. To provide the design method, some sufficient conditions are derived via applying Lyapunov function. According to discussions (Boyd et al., 1994; Boyd and Vandenberghe, 2004), one knows that convex optimal algorithm is an effective tool to solve optimal control problem involving LMI. Moreover, some further LMI problems were discussed and investigated by Scherer (2006) and Pipeleers et al. (2009). Referring to (Pipeleers et al., 2009), an extended LMI form provides less conservative property than standard LMI form in searching sufficient solutions with applying the convex optimal algorithm. Therefore, the derived conditions are converted into extended LMI form to design a mixed H_2/*Passivity* performance controller to guarantee the minimum H_2 performance under the desired passivity of linear stochastic systems. At last, some simulation results are used to show effectiveness and applicability of the proposed design method.

II. SYSTEM DESCRIPTIONS AND PROBLEM STATEMENTS

In this paper, a discrete-time linear stochastic system is described as follows:

$$x(k+1) = \mathbf{A}x(k) + \mathbf{B}u(k) + \mathbf{E}v(k) + \left(\bar{\mathbf{A}}x(k) + \bar{\mathbf{B}}u(k) + \bar{\mathbf{E}}v(k)\right)w(k) \tag{1a}$$

$$y(k) = \mathbf{C}_1 x(k) + \mathbf{D}_1 v(k) \tag{1b}$$

$$z(k) = \mathbf{C}_2 x(k) + \mathbf{D}_2 u(k) \tag{1c}$$

where $x(k) \in R^n$ is the state vector, $y(k) \in R^m$ is the measured output vector, $z(k) \in R^p$ is the controlled output vector, $u(k) \in R^q$ is the control input vector, $v(k) \in R^m$ is the disturbance input vector, and $w(k)$ is a scalar discrete type Brownian motion. $\mathbf{A}$, $\bar{\mathbf{A}}$, $\mathbf{B}$, $\bar{\mathbf{B}}$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{E}$ and $\bar{\mathbf{E}}$ are constant matrices with compatible dimensions. Referring to (Karatzas and Shreve, 1991), one can find the independent increment properties of $w(k)$, such as $E\{w(k)\} = 0$, $E\{w(k)\, w(k)\} = 0$ and $E\{w(k)\, w(k)\} = 1$, where $E\{\bullet\}$ denotes the expected value of $\bullet$. Besides, it should be noted that the pair ($\mathbf{A}$, $\mathbf{B}$) is known and controllable.

For dealing with stabilization problem of (1), the following state feedback controller is considered in this paper.

$$u(k) = \mathbf{F}x(k) \tag{2}$$

where $\mathbf{F} \in \mathrm{R}^{q\times n}$ is a feedback gain and is needed to be designed. Substituting (2) into (1), the following closed-loop system is inferred.

$$x(k+1) = \mathbf{A}_f x(k) + \mathbf{E}v(k) + \left(\bar{\mathbf{A}}_f x(k) + \bar{\mathbf{E}}v(k)\right)w(k) \tag{3a}$$

$$y(k) = \mathbf{C}_1 x(k) + \mathbf{D}_1 v(k) \tag{3b}$$

$$z(k) = \mathbf{C}_{2f} x(k) \tag{3c}$$

where $\mathbf{A}_f = \mathbf{A} + \mathbf{BF}$, $\bar{\mathbf{A}}_f = \bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{F}$ and $\mathbf{C}_{2f} = \mathbf{C}_2 + \mathbf{D}_2\mathbf{F}$.

In this paper, the passivity theory is substituted for the H_∞ scheme to constrain the effect of external disturbance on the closed-loop system (3). Referring to Lozano et al. (2000), the passivity theory can be introduced in the following definition.

Definition 1: The closed-loop system (3) with disturbance input $v(k)$ and measured output $y(k)$ is called passive if there exists matrices $\mathbf{S}_1$, $\mathbf{S}_2 \geq 0$ and $\mathbf{S}_3$ such that

$$E\left\{2\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)\right\} > E\left\{\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)\mathbf{S}_2 y(k) + \sum_{k=0}^{k_p} v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)\right\} \tag{4}$$

for any terminal time $k_p > 0$. #

From (Ku et al., 2010), power supply function (4) can be reduced into several cases for dealing with attenuation performance of the closed-loop system (3). For example, inequality (4) is reduced as a) H_∞ performance by setting $\mathbf{S}_1 = 0$, $\mathbf{S}_2 = \mathbf{I}$ and $\mathbf{S}_3 = -\gamma^2$ with positive scalar γ, b) positive real performance by setting $\mathbf{S}_1 = \mathbf{I}$, $\mathbf{S}_2 = 0$ and $\mathbf{S}_3 = 0$; c) strictly input passive performance by setting $\mathbf{S}_1 = \mathbf{I}$, $\mathbf{S}_2 = 0$ and $\mathbf{S}_3 = \gamma\mathbf{I}$ with positive scalar γ, d) strictly output passive performance by setting $\mathbf{S}_1 = \mathbf{I}$, $\mathbf{S}_2 = \gamma\mathbf{I}$ and $\mathbf{S}_3 = 0$ with positive scalar γ, e) strictly vary passive performance by setting $\mathbf{S}_1 = \mathbf{I}$, $\mathbf{S}_2 = \gamma\mathbf{I}$ and $\mathbf{S}_2 = \varsigma\mathbf{I}$ with positive scalars γ and ς. Thus, the passivity theory in Definition 1 proposes a general and flexible attenuation performance index. Besides, in case of $v(k) = 0$, the H_2 performance index defined in the following definition is applied to minimize output energy.

Definition 2 (Kim, 2001): The controller (2) is an H_2 performance measure for the closed-loop system (3) with $v(k) = 0$, if one can find a $\alpha > 0$ to satisfy the following inequality.

$$E\left\{\sum_{k=0}^{T_f} z^{\mathrm{T}}(k)\, z(k)\right\} < \alpha \tag{5}$$

where $T_f > 0$ is terminal time of control. #

Remark 1

Based on the definitions of this paper, the mixed performance of the closed-loop system (3) is dealt with achieving H_2 performance and passivity. With setting $\mathbf{S}_1 = 0$, $\mathbf{S}_2 = \mathbf{I}$ and $\mathbf{S}_3 = -\gamma^2$, the proposed mixed *H_2/Passivity* performance criterion can be reduced as the mixed H_2/H_∞ performance criterion. In addition, the control issue discussed in this paper is more general than the design method of Fioravanti et al. (2014) according to the consideration of stochastic behavior. Concluding the above description, the proposed design method is more general and flexible than the existing design methods (Kim, 2001; Yang et al., 2002; Fioravanti et al., 2014; Orihuela et al., 2015). #

Through Definition 1 and Definition 2, two goals are concerned to guarantee the required control performance. One of the goals is to ensure the attenuation performance of the closed-loop system (3). Another goal, in case of $v(k) = 0$, is to minimize output energy of the closed-loop system (3) with initial conditions. According to the above illustrations, the mixed *H_2/Passivity* performance control problem of the closed-loop system (3) is discussed in the next section.

III. MIXED H_2/PASSIVITY PERFORMANCE CONTROLLER DESIGN METHOD

In this section, some sufficient conditions are derived by Lyapunov function to achieve the above definitions. Through solving the sufficient conditions, some feasible solutions can be obtained to establish mixed *H_2/Passivity* performance controller (2) to stabilize closed-loop system (3).

Theorem 1

Given matrices $\mathbf{S}_1$, $\mathbf{S}_2 \geq 0$ and $\mathbf{S}_3$, if there exists a minimized positive scalar α, positive definite matrix $\mathbf{P}$ and feedback gain $\mathbf{F}$ satisfying the following inequalities then the asymptotical stability and mixed *H_2/Passivity* performance of the closed-loop system (3) are achieved in the mean square.

$$\begin{bmatrix} \mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f + \bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f - \mathbf{P} + \mathbf{C}_1^{\mathrm{T}}\mathbf{S}_2\mathbf{C}_1 & * \\ \mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{A}_f + \bar{\mathbf{E}}^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f - \mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1 + \mathbf{D}_1^{\mathrm{T}}\mathbf{S}_2\mathbf{C}_1 & \mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{E} + \bar{\mathbf{E}}^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}} + \mathbf{S}_3 - \mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1 - \mathbf{S}_1^{\mathrm{T}}\mathbf{D}_1 + \mathbf{D}_1^{\mathrm{T}}\mathbf{S}_2\mathbf{D}_1 \end{bmatrix} < 0 \tag{6}$$

$$\mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f + \bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f - \mathbf{P} + \mathbf{C}_{2f}^{\mathrm{T}}\mathbf{C}_{2f} < 0 \tag{7}$$

$$x^{\mathrm{T}}(0)\mathbf{P}x(0) - \alpha < 0 \tag{8}$$

where * denotes the transposed elements or matrices for symmetric position.

Proof:

Let us choose the following Lyapunov function.

$$V(x(k)) = x^{\mathrm{T}}(k)\mathbf{P}x(k) \tag{9}$$

Taking the first forward difference of (9), one has

$$\begin{aligned} \Delta V(x(k)) &= x^{\mathrm{T}}(k+1)\mathbf{P}x(k+1) - x^{\mathrm{T}}(k)\mathbf{P}x(k) \\ &= \left(\mathbf{A}_f x(k) + \mathbf{E}v(k) + \left(\bar{\mathbf{A}}_f x(k) + \bar{\mathbf{E}}v(k)\right)w(k)\right)^{\mathrm{T}} \mathbf{P} \\ &\quad \times \left(\mathbf{A}_f x(k) + \mathbf{E}v(k) + \left(\bar{\mathbf{A}}_f x(k) + \bar{\mathbf{E}}v(k)\right)w(k)\right) \\ &\quad - x^{\mathrm{T}}(k)\mathbf{P}x(k) \end{aligned} \tag{10}$$

Taking expectation of (10), the following equation can be obtained with the independent increment property of Brownian motion (Karatzas and Shreve, 1991).

$$\begin{aligned} &E\{\Delta V(x(k))\} \\ &= E\left\{x^{\mathrm{T}}(k+1)\mathbf{P}x(k+1) - x^{\mathrm{T}}(k)\mathbf{P}x(k)\right\} \\ &= E\left\{\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f + \bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f - \mathbf{P} & * \\ \mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{A}_f + \bar{\mathbf{E}}^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f & \mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{E} + \bar{\mathbf{E}}^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}} \end{bmatrix} \right. \\ &\qquad \left. \times \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}\right\} \end{aligned} \tag{11}$$

Let us define the following cost function with zero initial condition.

$$\begin{aligned}\Gamma(x,v,k) &= E\left\{\sum_{k=0}^{k_p}\left(y^{\mathrm{T}}(k)\mathbf{S}_2 y(k)+v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)\right.\right.\\ &\qquad \left.\left.-2y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)\right)\right\}\\ &= E\left\{\sum_{k=0}^{k_p}\left(y^{\mathrm{T}}(k)\mathbf{S}_2 y(k)+v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)\right.\right.\\ &\qquad -2y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)\\ &\qquad \left.\left.+\Delta V(x(k))\right)-V(x(k_p+1))\right\}\\ &\le E\left\{\sum_{k=0}^{k_p}\left(y^{\mathrm{T}}(k)\mathbf{S}_2 y(k)+v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)\right.\right.\\ &\qquad \left.\left.-2y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)+\Delta V(x(k))\right)\right\}\\ &= E\left\{\sum_{k=0}^{k_p}\Psi(x,v,k)\right\}\end{aligned} \tag{12}$$

where

$$\begin{aligned}\Psi(x,v,k) &= y^{\mathrm{T}}(k)\mathbf{S}_2 y(k)+v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)\\ &\qquad -2y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)+\Delta V(x(k))\end{aligned} \tag{13}$$

Substituting (3b) and (11) into (13), one has

$$\Psi(x,v,k)=\begin{bmatrix}x(k)\\ v(k)\end{bmatrix}^{\mathrm{T}}\mathbf{\Lambda}\begin{bmatrix}x(k)\\ v(k)\end{bmatrix} \tag{14}$$

where

$$\mathbf{\Lambda}=\begin{bmatrix}\mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f+\bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f-\mathbf{P}+\mathbf{C}_1^{\mathrm{T}}\mathbf{S}_2\mathbf{C}_1 & * \\ \mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{A}_f+\bar{\mathbf{E}}^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f-\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1+\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_2\mathbf{C}_1 & \mathbf{E}^{\mathrm{T}}\mathbf{P}\mathbf{E}+\bar{\mathbf{E}}^{\mathrm{T}}\mathbf{P}\bar{\mathbf{E}}+\mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1^{\mathrm{T}}\mathbf{D}_1+\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_2\mathbf{D}_1\end{bmatrix}. \tag{15}$$

Obviously, the left-hand side of condition (6) equals to $\mathbf{\Lambda}$. Thus, if condition (6) holds, then (15) is negative according to $\mathbf{\Lambda} < 0$. Moreover, $\mathbf{\Lambda} < 0$ implies $\Psi(x, v, k) < 0$ from (14). Due to $\Psi(x, v, k) < 0$, one can find $\Gamma(x, v, k) < 0$ from (12). And then, the following inequality can be easily found.

$$\begin{aligned}&E\left\{2\sum_{k=0}^{k_p}y^{T}(k)\mathbf{S}_1 v(k)\right\}\\ &> E\left\{\sum_{k=0}^{k_p}y^{T}(k)\mathbf{S}_2 y(k)+\sum_{k=0}^{k_p}v^{T}(k)\mathbf{S}_3 v(k)\right\}\end{aligned} \tag{16}$$

Because (16) is equivalent to (4), the closed-loop system (3) is passive with the given $\mathbf{S}_1$, $\mathbf{S}_2 \ge 0$ and $\mathbf{S}_3$. Next, the asymptotical stability of the closed-loop system (3) with zero disturbance input is analyzed in the following derivative. When the external disturbance is zero, i.e., $v(k) = 0$, Eq. (11) can be rewritten as the following relations.

$$\begin{aligned}&E\{\Delta V(x(k))\}\\ &= E\left\{x^{\mathrm{T}}(k)\left(\mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f+\bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f-\mathbf{P}\right)x(k)\right\}\end{aligned} \tag{17}$$

Introducing (3c) into (17), one has

$$\begin{aligned}&E\{\Delta V(x(k))\}\\ &= E\left\{x^{\mathrm{T}}(k)\left(\mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f+\bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f-\mathbf{P}\right)x(k)\right.\\ &\quad \left.+z^{\mathrm{T}}(k)z(k)-z^{\mathrm{T}}(k)z(k)\right\}\\ &= E\left\{x^{\mathrm{T}}(k)\left(\mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f+\bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f-\mathbf{P}+\mathbf{C}_{2f}^{\mathrm{T}}\mathbf{C}_{2f}\right)x(k)\right.\\ &\quad \left.-z^{\mathrm{T}}(k)z(k)\right\}\\ &< E\left\{x^{\mathrm{T}}(k)\left(\mathbf{A}_f^{\mathrm{T}}\mathbf{P}\mathbf{A}_f+\bar{\mathbf{A}}_f^{\mathrm{T}}\mathbf{P}\bar{\mathbf{A}}_f-\mathbf{P}+\mathbf{C}_{2f}^{\mathrm{T}}\mathbf{C}_{2f}\right)x(k)\right\}\end{aligned}. \tag{18}$$

If inequality (7) holds, then one can find $E\{\Delta V(x(k))\} < 0$ from (18). Referring to (Ghaoui, 1995) and $E\{\Delta V(x(k))\} < 0$, the closed-loop system (3) with zero external disturbance input is asymptotically stable in the mean square. Besides, condition (7) implies the following inequality.

$$E\left\{\Delta V(x(k))+z^{\mathrm{T}}(k)z(k)\right\}<0 \tag{19}$$

and

$$E\{\Delta V(x(k))\}<E\left\{-z^{\mathrm{T}}(k)z(k)\right\} \tag{20}$$

Summarizing (20) from 0 to T_f, we have

$$E\left\{x^{\mathrm{T}}(T_f)\mathbf{P}x(T_f)-x^{\mathrm{T}}(0)\mathbf{P}x(0)\right\}<E\left\{-\sum_{k=0}^{T_f}z^{\mathrm{T}}(k)\,z(k)\right\}. \tag{21}$$

Because condition (7) holds, one knows that the closed-loop system (3) is asymptotically stable, that implies $x(T_f) \to 0$ and $x^{\mathrm{T}}(T_f)\mathbf{P}x(T_f) \to 0$ as $T_f \to \infty$. Therefore, inequality (21) can be furtherly inferred as the following inequality.

$$E\left\{\sum_{k=0}^{T_f} z^{\mathrm{T}}(k)\, z(k)\right\} < E\left\{x^{\mathrm{T}}(0)\mathbf{P}x(0)\right\} \tag{22}$$

From inequality (22), it is known that $E\left\{x^{\mathrm{T}}(0)\mathbf{P}x(0)\right\}$ is the upper bound of output energy. If condition (8) holds, then the following inequalities can be obtained due to $E\{a\} = a$ where a is scalar.

$$E\left\{x^{\mathrm{T}}(0)\mathbf{P}x(0) - \alpha\right\} \le 0 \tag{23}$$

and

$$E\left\{x^{\mathrm{T}}(0)\mathbf{P}x(0)\right\} \le \alpha \tag{24}$$

From (22) and (24), the following relation can be directly found.

$$E\left\{\sum_{k=0}^{T_f} z^{\mathrm{T}}(k)\, z(k)\right\} < \alpha \tag{25}$$

Because (25) is equivalent to (5), one knows that if conditions (7) and (8) of this theorem are satisfied, then the H_2 performance of the closed-loop system (3) is achieved. Moreover, the output energy is minimized subject to α. The proof of this theorem is completed. #

In Theorem 1, the sufficient conditions are derived by using Lyapunov function to discuss the asymptotical stability and mixed H_2/*Passivity* performance of (3) in the mean square. To apply convex optimal algorithm (Boyd and Vandenberghe, 2004), the sufficient conditions in Theorem 1 are converted into LMI form in the following theorem.

Theorem 2

For the given matrices $\mathbf{S}_1$, $\mathbf{S}_2 \ge 0$ and $\mathbf{S}_3$, the asymptotical stability and mixed H_2/*Passivity* performance of closed-loop system (3) are achieved in the mean square if there exists positive scalar α, positive definite matrix $\mathbf{P}$, arbitrary matrix $\mathbf{G}$ and feedback gain $\mathbf{F}$ such that

$$\begin{bmatrix} \mathbf{X}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} & * & * & * & * \\ -\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{G} & \mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1\mathbf{D}_1 & * & * & * \\ \mathbf{AG}+\mathbf{BK} & \mathbf{E} & -\mathbf{X} & * & * \\ \overline{\mathbf{A}}\mathbf{G}+\overline{\mathbf{B}}\mathbf{K} & \overline{\mathbf{E}} & 0 & -\mathbf{X} & * \\ \mathbf{C}_1\mathbf{G} & \mathbf{D}_1 & 0 & 0 & -\mathbf{S}_2^{-1} \end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix} \mathbf{X}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} & * & * & * \\ \mathbf{C}_2\mathbf{G}+\mathbf{D}_2\mathbf{K} & -\mathbf{I} & * & * \\ \mathbf{AG}+\mathbf{BK} & 0 & -\mathbf{X} & * \\ \overline{\mathbf{A}}\mathbf{G}+\overline{\mathbf{B}}\mathbf{K} & 0 & 0 & -\mathbf{X} \end{bmatrix} < 0 \tag{27}$$

$$\begin{bmatrix} -\alpha & x^{\mathrm{T}}(0) \\ x(0) & -\mathbf{X} \end{bmatrix} < 0. \tag{28}$$

where $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{K} = \mathbf{FG}$.

Proof:

Applying Schur complement (Boyd et al., 1994), inequality (6) can be converted into the following inequality.

$$\begin{bmatrix} -\mathbf{P} & * & * & * & * \\ -\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1 & \mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1\mathbf{D}_1 & * & * & * \\ \mathbf{A}_f & \mathbf{E} & -\mathbf{P}^{-1} & * & * \\ \overline{\mathbf{A}}_f & \overline{\mathbf{E}} & 0 & -\mathbf{P}^{-1} & * \\ \mathbf{C}_1 & \mathbf{D}_1 & 0 & 0 & -\mathbf{S}_2^{-1} \end{bmatrix} < 0 \tag{29}$$

Multiplying the both side of (29) by $diag\left\{\mathbf{G}^{\mathrm{T}}, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}\right\}$ and $diag\left\{\mathbf{G}, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}\right\}$, where $diag\{\bullet\}$ denotes the diagonal matrix with element $\bullet$, one has

$$\begin{bmatrix} -\mathbf{G}^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{G} & * & * & * & * \\ -\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{G} & \mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1\mathbf{D}_1 & * & * & * \\ \mathbf{A}_f\mathbf{G} & \mathbf{E} & -\mathbf{P}^{-1} & * & * \\ \overline{\mathbf{A}}_f\mathbf{G} & \overline{\mathbf{E}} & 0 & -\mathbf{P}^{-1} & * \\ \mathbf{C}_1\mathbf{G} & \mathbf{D}_1 & 0 & 0 & -\mathbf{S}_2^{-1} \end{bmatrix} < 0\cdot \tag{30}$$

According to $\mathbf{P} > 0$, one holds the following fact.

$$\left(\mathbf{P}^{-1}-\mathbf{G}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{P}\left(\mathbf{P}^{-1}-\mathbf{G}\right) \ge 0 \tag{31}$$

Arranging (31) one can find the following relation to replacing the bilinear term in (30).

$$\mathbf{P}^{-1}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} \ge -\mathbf{G}^{\mathrm{T}}\mathbf{P}\,\mathbf{G} \tag{32}$$

Thus, one has

$$\begin{bmatrix} \mathbf{P}^{-1}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} & * & * & * & * \\ -\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{G} & \mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1\mathbf{D}_1 & * & * & * \\ \mathbf{A}_f\mathbf{G} & \mathbf{E} & -\mathbf{P}^{-1} & * & * \\ \overline{\mathbf{A}}_f\mathbf{G} & \overline{\mathbf{E}} & 0 & -\mathbf{P}^{-1} & * \\ \mathbf{C}_1\mathbf{G} & \mathbf{D}_1 & 0 & 0 & -\mathbf{S}_2^{-1} \end{bmatrix} < 0. \tag{33}$$

Substituting $\mathbf{A}_f = \mathbf{A} + \mathbf{BF}$ and $\bar{\mathbf{A}}_f = \bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{F}$ into (33), the following relation can be obtained by setting $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{K} = \mathbf{FG}$.

$$\begin{bmatrix} \mathbf{X}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} & * & * & * & * \\ -\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{G} & \mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1\mathbf{D}_1 & * & * & * \\ \mathbf{AG}+\mathbf{BK} & \mathbf{E} & -\mathbf{X} & * & * \\ \bar{\mathbf{A}}\mathbf{G}+\bar{\mathbf{B}}\mathbf{K} & \bar{\mathbf{E}} & 0 & -\mathbf{X} & * \\ \mathbf{C}_1\mathbf{G} & \mathbf{D}_1 & 0 & 0 & -\mathbf{S}_2^{-1} \end{bmatrix} < 0 \quad (34)$$

It is easy to find that (34) is equivalent to (26). Thus, if condition (26) holds, then condition (6) is satisfied. Besides, (27) and (28) can be respectively derived from conditions (7) and (8) with the above converting processes. Thus, the proofs of (28) and (29) are omitted here. The proof of Theorem 2 is complete. #

In Theorem 2, the sufficient conditions are converted into extended LMI form via introducing arbitrary matrix and applying transformation technologies. Therefore, the feasible solutions can be obtained via using convex optimization algorithm to establish controller (2) such that the H_2 performance and passivity of the closed-loop system (3) are achieved.

Remark 2

Referring to (Pipeleers et al., 2009), one can find that the extended LMI form possesses two advantages. One of the advantages is that the extended LMI form reduces the conservatism in finding feasible solutions for the conditions in Theorem 2. Another advantage is that the sufficient conditions in Theorem 2 can be reduced to standard LMI form by setting $\mathbf{G} = \mathbf{P}^{-1}$. #

By setting $\mathbf{S}_2 = 0$, a sufficient condition (26) in Theorem 2 becomes as non-standard LMI form that is difficult to be solved by the convex optimal algorithm. For the case as $\mathbf{S}_2 = 0$, the sufficient conditions in Theorem 2 can be rewritten as the following corollary.

Corollary 1

For the given matrices $\mathbf{S}_1$ and $\mathbf{S}_3$, the asymptotical stability and mixed *H_2/Passivity* performance of closed-loop system (3) are achieved in the mean square, if there exists positive scalar α, positive definite matrix $\mathbf{P}$, arbitrary matrix $\mathbf{G}$ and feedback gain $\mathbf{F}$ such that

$$\begin{bmatrix} \mathbf{X}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} & * & * & * \\ -\mathbf{S}_1^{\mathrm{T}}\mathbf{C}_1\mathbf{G} & \mathbf{S}_3-\mathbf{D}_1^{\mathrm{T}}\mathbf{S}_1-\mathbf{S}_1\mathbf{D}_1 & * & * \\ \mathbf{AG}+\mathbf{BK} & \mathbf{E} & -\mathbf{X} & * \\ \bar{\mathbf{A}}\mathbf{G}+\bar{\mathbf{B}}\mathbf{K} & \bar{\mathbf{E}} & 0 & -\mathbf{X} \end{bmatrix} < 0 \quad (35)$$

$$\begin{bmatrix} \mathbf{X}-\mathbf{G}^{\mathrm{T}}-\mathbf{G} & * & * & * \\ \mathbf{C}_2\mathbf{G}+\mathbf{D}_2\mathbf{K} & -\mathbf{I} & * & * \\ \mathbf{AG}+\mathbf{BK} & 0 & -\mathbf{X} & * \\ \bar{\mathbf{A}}\mathbf{G}+\bar{\mathbf{B}}\mathbf{K} & 0 & 0 & -\mathbf{X} \end{bmatrix} < 0 \quad (36)$$

$$\begin{bmatrix} -\alpha & x^{\mathrm{T}}(0) \\ x(0) & -\mathbf{X} \end{bmatrix} < 0 . \quad (37)$$

Proof:

Following the proof procedure of Theorem 2 and setting $\mathbf{S}_2 = 0$, the proof of this theorem can be easily obtained and it is omitted here. #

Based on Corollary 1, one can also apply convex optimal algorithm to find feasible solutions to establish controller (2) for the closed-loop system (3). For demonstrating effectiveness and usefulness of the proposed design method, some numerical simulations are proposed in the following section.

IV. SIMULATION

In this section, two numerical examples are proposed. In the first example, a dissipative controller design method (Tan et al., 2010) is applied to compare with the proposed design method. Through the simulation results in the first example, one can find that the considerations of stochastic behavior and H_2 performance are important issues in practical control problem. In the second example, a mixed performance control problem of an inverted pendulum on a cart system is discussed and solved by the proposed design method. Moreover, a comparison between the proposed design method and a mixed H_2/H_∞ performance design method (Fioravanti et al., 2014) is provided in Example 2. Through this example, one can find that the proposed design method can be reduced into the design method (Fioravanti et al., 2014). And, the proposed design method possesses less conservatism than the method (Fioravanti et al., 2014) based on the simulation results.

Example 1

Referring to (Tanaka and Sano, 1994), the nonlinear dynamic equations of truck-trailer system can be proposed as follows:

$$\begin{aligned} x_1(k+1) &= 1.363x_1(k) - 0.7143u(k) + 0.1v(k) \\ &\quad + \big(0.09x_1(k) - 0.45u(k) \\ &\quad -0.01v(k)\big)w(k) \end{aligned} \quad (38a)$$

$$x_2(k+1) = -0.363x_1(k) + x_2(k) + \big(0.1x_2(k)\big)w(k) \quad (38b)$$

$$\begin{aligned} x_3(k+1) &= -2\times\sin\big(-0.18x_1(k)+x_2(k)\big) + x_3(k) \\ &\quad + \big(0.2x_3(k)\big)w(k) \end{aligned} \quad (38c)$$

$$y(k) = x_1(k) + v(k) \quad (38d)$$

where $x_1(k)$ is the angle difference truck and trailer, $x_2(k)$ is the angle of the trailer, and $x_3(k)$ is the vertical position of the rear

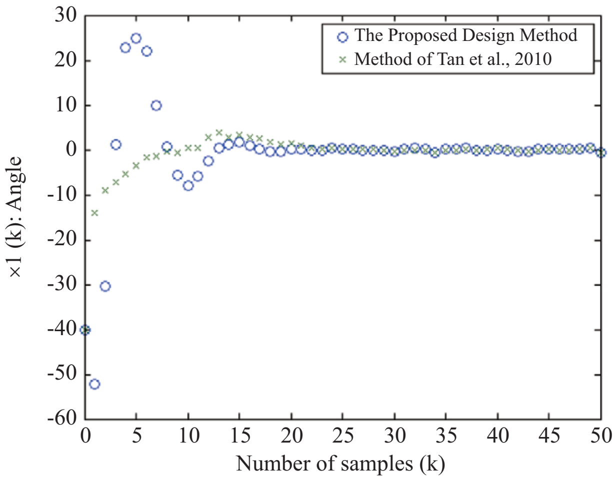

Fig. 1. Responses of state $x_1(k)$ in Example 1.

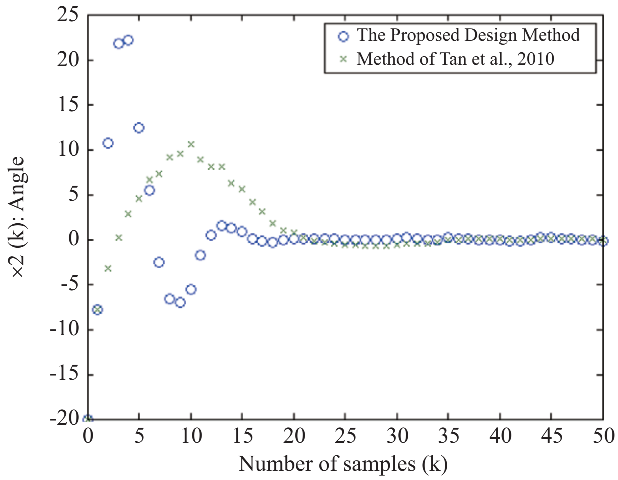

Fig. 2. Responses of state $x_2(k)$ in Example 1.

end of the trailer. Moreover, the external disturbance $v(k)$ is zero-mean white noise with variance 0.1. Besides, the following linear model can be obtained to represent the local behavior of (38) around the equilibrium point $x^{ep}(k)=\begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. Moreover, the output $z(k)$ is added to achieve H_2 performance.

$$x(k+1)=\mathbf{A}x(k)+\mathbf{B}u(k)+\mathbf{E}v(k)+\bar{\mathbf{A}}x(k)w(k) \quad \text{(39a)}$$

$$y(k)=\mathbf{C}_1x(k)+\mathbf{D}_1v(k) \quad \text{(39b)}$$

$$z(k)=\mathbf{C}_2x(k)+\mathbf{D}_2u(k) \quad \text{(39c)}$$

where $\mathbf{A}=\begin{bmatrix}1.3636 & 0 & 0\\ -0.3636 & 1 & 0\\ 0.3636 & -2 & 1\end{bmatrix}$, $\bar{\mathbf{A}}=\begin{bmatrix}0.09 & 0 & 0\\ 0 & 0.1 & 0\\ 0 & 0 & 0.2\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}-0.7143\\ 0\\ 0\end{bmatrix}$, $\mathbf{C}_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$, $\mathbf{C}_2=\begin{bmatrix}0 & 0 & 1\end{bmatrix}$, $\mathbf{D}_1$ = $\mathbf{D}_2$ = 1 and $\mathbf{E}=\begin{bmatrix}0.01 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. In order to apply the proposed design method, the $\mathbf{S}_1$ = 1, $\mathbf{S}_2$ = 0.8 and $\mathbf{S}_3$ = 0.8, initial condition $x(0)=\begin{bmatrix}-40^\circ & -20^\circ & 1\end{bmatrix}^{\mathrm{T}}$ and sampled period as 2 second are determined. And then, the following controller can be established via solving the sufficient conditions in Theorem 2.

$$u(k)=\mathbf{F}x(k) \quad (40)$$

where $\mathbf{F}=\begin{bmatrix}2.6878 & -3.3552 & 0.5516\end{bmatrix}$. Moreover, the minimum value of $\alpha = 4.9961$ is also obtained. Applying (40), the responses of (38) are stated in Figs. 1-3 with the initial condition. Besides, the achievement of performances can be checked

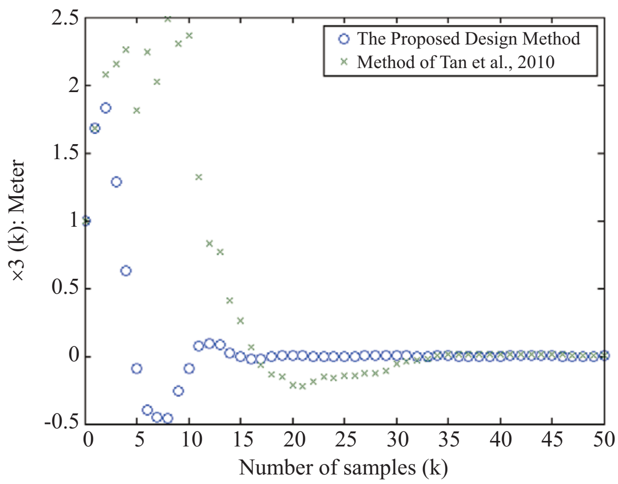

Fig. 3. Responses of state $x_3(k)$ in Example 1.

by the following equations with the simulation results.

$$\frac{2\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)}{\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)\mathbf{S}_2 y(k)+\sum_{k=0}^{k_p} v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)}=1.006 \quad (41)$$

and

$$\sum_{k=0}^{k_p} z^{\mathrm{T}}(k)z(k)=3.156 \quad (42)$$

According to the above equations, the value of (41) is bigger than one that satisfies Definition 1. Moreover, the value in (42) is smaller than the obtained minimum α that achieves Definition 2. From (41), (42) and Figs. 1-3, the asymptotical stability and mixed H_2/*Passivity* performance of truck-trailer system (38) are thus achieved by the controller (40).

Referring to (Tan et al., 2010), a dissipative controller design method was proposed without considering H_2 performance and stochastic behaviors. Based on the same matrices $\mathbf{S}_1 = 1$, $\mathbf{S}_2 = 0.8$ and $\mathbf{S}_3 = 0.8$, the following controller can be designed by using the method (Tan et al., 2010).

$$u(k) = \mathbf{F}x(k) \tag{43}$$

where $\mathbf{F} = \begin{bmatrix} 2.2473 & -1.1462 & 0.0806 \end{bmatrix}$. With the same initial condition, the responses of (38) driven by (43) are also proposed in Figs. 1-3. Based on those responses, the following values can be obtained.

$$\frac{2\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)\mathbf{S}_1 v(k)}{\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)\mathbf{S}_2 y(k) + \sum_{k=0}^{k_p} v^{\mathrm{T}}(k)\mathbf{S}_3 v(k)} = 1.165 \tag{44}$$

and

$$\sum_{k=0}^{k_p} z^{\mathrm{T}}(k) z(k) = 47 \tag{45}$$

Because the value in (44) is bigger than one, the passivity performance of (38) driven by (43) is achieved. However, the value in (45) is bigger than the obtained $\alpha = 4.9961$. Thus, the H_2 performance of (38) driven by (43) is not achieved. It means that the output energy of (38) driven by (43) is bigger than that driven by (40).

Referring to Figs. 1-3, controller (40) possesses better settling time than controller (43). Besides, the overshoot of (38) driven by (43) is bigger than that driven by (40). The poor control performance of (43) is caused by the considered stochastic behavior. Moreover, because the H_2 performance was not concerned by Tan et al. (2010), the response of $x_3(k)$ cannot be constrained under the given requirement. Concluding this simulation results, the proposed design method provides better control performance than the method of Tan et al. (2010).

Example 2

In this example, a comparison between the proposed design method and the method (Fioravanti et al., 2014) is provided to show advantages of this paper. From (Fioravanti et al., 2014), a mixed H_2/H_∞ performance controller design method has been proposed for discrete-time linear systems. In this example, two cases are proposed to emphasize the contribution of this paper. One of the cases is to show that the proposed design method provides less conservative than the method (Fioravanti et al., 2014). Another case is to emphasize the importance of considering stochastic behavior under the same performance indexes. Referring to (Iordanou and Surgenor, 1997), the following discrete-time inverted pendulum on a cart system was modeled with sampling time as 0.01 second. To apply the proposed design method, an external disturbance $v(k)$ and a multiplicative noise $w(k)$ are added.

$$\begin{aligned} x(k+1) = {} & \mathbf{A}x(k) + \mathbf{B}u(k) + \mathbf{E}v(k) \\ & + \left(\bar{\mathbf{A}}x(k) + \bar{\mathbf{B}}u(k) + \bar{\mathbf{E}}v(k)\right)w(k) \end{aligned} \tag{46a}$$

$$y(k) = \mathbf{C}_1 x(k) + \mathbf{D}_1 v(k) \tag{46b}$$

$$z(k) = \mathbf{C}_2 x(k) + \mathbf{D}_2 u(k) \tag{46c}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) & x_4(k) \end{bmatrix}^{\mathrm{T}}$, $x_1(k)$ is the cart position, $x_2(k)$ is the cart velocity, $x_3(k)$ is the payload angle, $x_4(k)$ is the payload angle velocity, $u(k)$ is the applied force, and $v(k)$ is the zero-mean white noise with unit variance. And, system matrices are presented by $\mathbf{E} = \begin{bmatrix} 0 & 0 & 0.01 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{C}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$, $\mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{D}_1 = \mathbf{D}_2 = 1$, $\mathbf{A} = \begin{bmatrix} 1 & 0.0087 & 0 & 0 \\ 0 & 0.7515 & 0 & 0 \\ 0 & -0.0111 & 1.0015 & 0.0100 \\ 0 & -2.1235 & 0.3052 & 0.9999 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0.0027 \\ 0.5219 \\ 0.0234 \\ 4.4593 \end{bmatrix}$. In order to compare with the method (Fioravanti et al., 2014), the matrices in Definition 1 are set as $\mathbf{S}_1 = 0$, $\mathbf{S}_2 = \mathbf{I}$ and $\mathbf{S}_3 = -\gamma^2$ such that the proposed design method is reduced as mixed H_2/H_∞ performance controller design method. And, the initial condition $x(0) = \begin{bmatrix} 0.5 & 0 & 50^\circ & 0 \end{bmatrix}^{\mathrm{T}}$ is assumed in the following cases.

Case 1

In the first case, the multiplicative noise terms in (46a) are set as zero ($\bar{\mathbf{A}} = \bar{\mathbf{B}} = \bar{\mathbf{E}} = 0$). Moreover, the attenuation level γ and upper bounded α are respectively fixed to apply the proposed design method and the method (Fioravanti et al., 2014) to solve the mixed performance control problem of (46). Through MATLAB LMI Toolbox, simulation results of this case are concluded in Table 1. From Table 1, by fixing γ, the value as $\alpha = 30.4778$ can be obtained by the proposed design method. And, the value of $\alpha = 32.0344$ is obtained by applying the method (Fioravanti et al., 2014). It is easy to see that the smaller upper bound α can be found by using the proposed design method than one found by Fioravanti et al. (2014). Besides, under fixing α, the attenuation level $\gamma = 1.2093$ is obtained by the proposed design method. Moreover, an attenuation level $\gamma = 1.8334$ is obtained by the method (Fioravanti et al., 2014). It should be noted that the proposed design method provides smaller attenuation level than the method (Fioravanti et al., 2014). It means that the proposed design method provides better attenuation performance than the method (Fioravanti

Table 1. Comparison Between the Proposed Design Method and Method of (Fioravanti et al., 2014).

Method	Fixing $\gamma = 1.1$	Fixing $\alpha = 20$
The Proposed Design Method	$\alpha = 30.4778$	$\gamma = 1.2093$
Method of (Fioravanti et al., 2014)	$\alpha = 32.0344$	$\gamma = 1.8334$

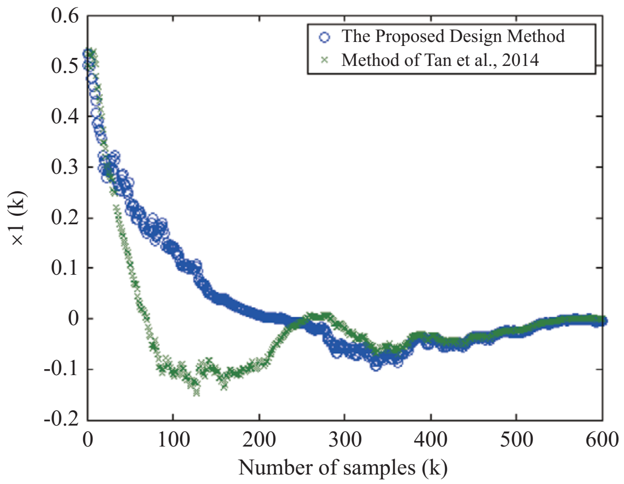

Fig. 4. Responses of state $x_1(k)$ in Example 2.

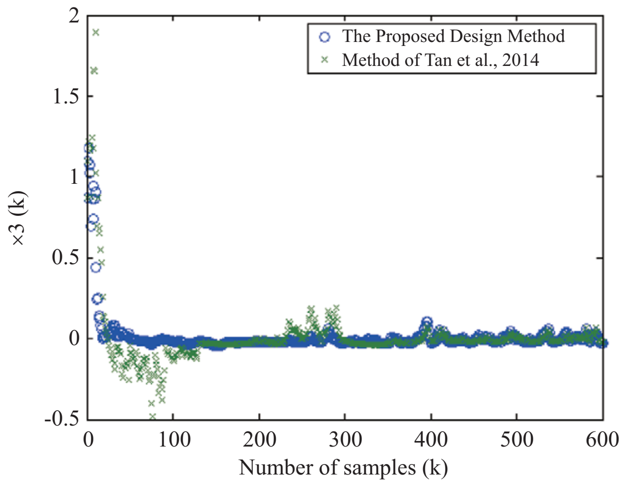

Fig. 6. Responses of state $x_3(k)$ in Example 2.

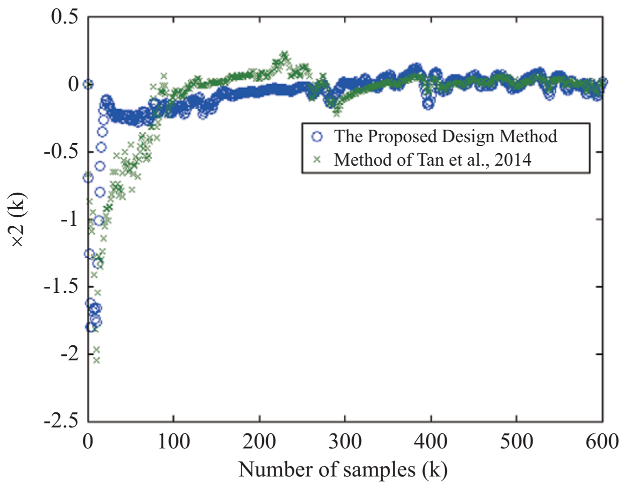

Fig. 5. Responses of state $x_2(k)$ in Example 2.

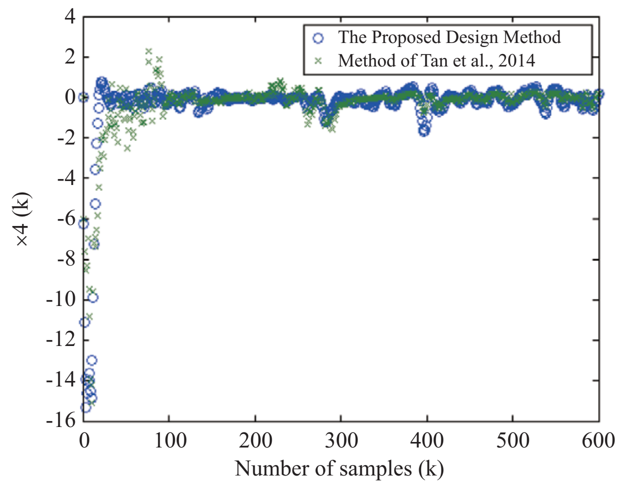

Fig. 7. Responses of state $x_4(k)$ in Example 2.

et al., 2014). Based on those simulation results, one can find that the conservatism of the proposed design method is less than one of the method (Fioravanti et al., 2014) according to extended LMI form. Therefore, the proposed design method is less conservative than the method (Fioravanti et al., 2014).

Case 2

In this case, the matrices of multiplicative noise term of (46) are set as $\bar{\mathbf{A}} = \begin{bmatrix} 0.07 & 0.01 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0.001 & 0.3 & 0 \\ 0 & 0.1 & 0 & 0.01 \end{bmatrix}$, $\bar{\mathbf{B}} = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.05 \\ 0.03 \end{bmatrix}$ and $\bar{\mathbf{E}} = \begin{bmatrix} 0 & 0 & 0.001 & 0 \end{bmatrix}^T$. Moreover, minimum upper bound $\alpha = 20$ and attenuation level $\gamma = 2$ are simultaneously fixed. Applying the proposed design method, the following controller can be established.

$$u(k) = \mathbf{F}x(k) \tag{47}$$

where $\mathbf{F} = \begin{bmatrix} 1.0165 & 1.4476 & -1.5815 & -0.3116 \end{bmatrix}$. Applying (47), the responses of (46) are stated in Figs. 4-7. Based on those responses, the following values can be obtained to check achievements of Definition 1 and Definition 2, respectively.

$$E\left\{\frac{\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)y(k)}{\sum_{k=0}^{k_p} v^{\mathrm{T}}(k)v(k)}\right\}=1.026 \tag{48}$$

and

$$E\left\{\sum_{k=0}^{k_p} z^{\mathrm{T}}(k)z(k)\right\}=6.212 \tag{49}$$

From the above results, one can find that the value of (48) is smaller than the given attenuation level $\gamma = 2$. According to Definition 1, the H_∞ performance of the system (46) driven by (47) is achieved. Besides, the value in (49) is smaller than the given $\alpha = 25$. Thus, the H_2 performance of (46) driven by (47) is achieved.

In addition, the following controller can be designed by using the method (Fioravanti et al., 2014) with the same $\alpha = 20$ and $\gamma = 2$.

$$u(k)=\mathbf{F}x(k) \tag{50}$$

where $\mathbf{F}=\begin{bmatrix}0.9777 & 1.2568 & -1.7609 & 0.2871\end{bmatrix}$. Based on (50), responses of (46) are stated in Figs. 4-7. Also, the following equations are proposed to ensure achievement of Definition 1 and Definition 2.

$$E\left\{\frac{\sum_{k=0}^{k_p} y^{\mathrm{T}}(k)y(k)}{\sum_{k=0}^{k_p} v^{\mathrm{T}}(k)v(k)}\right\}=1.054 \tag{51}$$

and

$$E\left\{\sum_{k=0}^{k_p} z^{\mathrm{T}}(k)z(k)\right\}=8.507 \tag{52}$$

From (51) and (52), one knows that the H_2 and H_∞ performances of (46) are achieved.

Referring to the above simulation results, asymptotical stability and mixed H_2/H_∞ performance of (46) can be achieved by using (47) or (50). However, from Figs. 4-7, one can easily find that the overshoot of (46) driven by (47) is smaller than that driven by (50). Moreover, the settling time of (46) driven by (47) is short than that driven by (50). Those poor performances in controller (50) designed by Fioravanti et al. (2014) are caused via the stochastic behaviors. According to the simulation results, it can be thus concluded that the proposed mixed H_2/*Passivity* controller design method proposes some improvements for (Fioravanti et al., 2014) in stabilizing the discrete-time linear stochastic systems.

Concluding the simulation results of this section, the consideration of stochastic behavior is an important issue in practical control because it always causes poor performance during control process. Besides, the derived sufficient conditions in Theorem 2 provides some relaxations in searching feasible solutions according to an arbitrary matrix $\mathbf{G}$. Therefore, the proposed design method is not only less conservative but also more general than the methods (Tan et al., 2010; Fioravanti et al., 2014).

V. CONCLUSIONS

In this paper, a general and flexible mixed performance control problem of the discrete-time linear stochastic systems has been discussed by H_2 control scheme and passivity theory to achieve minimized output energy and external disturbance constraint. To proposed mixed H_2/*Passivity* performance controller design method, some sufficient conditions were derived by Lyapunov function. Moreover, the derived sufficient conditions were converted into extended LMI form to reduce some conservatism in searching feasible solutions. Therefore, one can find the feasible solutions to build a controller such that the asymptotical stability and mixed H_2/*Passivity* performance of the stochastic system are achieved in the mean square. Finally, some simulation results have been proposed to show the effectiveness and applicability of the proposed design method.

ACKNOWLEDGEMENTS

This work was supported by the Ministry of Science and Technology of the Republic of China under Contract No. MOST 104-2221-E-019-015.

REFERENCES

Boyd, S. and L. Vandenberghe (2004). Convex Optimization. Cambridge University Press: Cambridge, UK.

Boyd, S., L. E. Ghaoui, E. Feron and V. Balakrishnan (1994). Linear Matrix Inequalities in System and Control Theory. SIAM: Philadelphia, PA.

Chung, H. Y. and W. J. Chang (1990). Input and state covariance control for bilinear stochastic discrete systems. C-TAT, Control-Theory and Advanced Technology 6, 655-667.

Du, D. (2006). Generalized H_2 output feedback controller design for uncertain discrete-time switched systems via switched Lyapunov functions. Nonlinear Analysis: Theory, Methods and Applications 65, 2135-2146.

Fioravanti, A. R., A. P. C. Goncalves and J. C. Geromel (2014). Optimal H_2 and H_∞ mode-independent control or generalized Bernoulli jump systems. ASME, Journal of Dynamic Systems Measurement and Control 136, 011004-1-011004-6.

Ghaoui, L. E. (1995). State-feedback control of systems with multiplicative noise via linear matrix inequalities. Systems and Control Letters 24, 223-228.

Iordanou, H. N. and B. W. Surgenor (1997). Experimental evaluation of the robustness of discrete sliding mode control versus linear quadratic control. IEEE Trans. on Control Systems Technology 5, 254-260.

Jiang, Z. P. and D. J. Hill (1998). Passivity and disturbance attenuation via output feedback for uncertain nonlinear systems. IEEE Trans. on Automatic Control 43, 992-997.

Karatzas, I. and S. E. Shreve (1991). Brownian Motion and Stochastic Cal-

culus. Springer: New York.
Kim, J. H. (2001). Robust mixed H_2/H_∞ control of time-varying delay systems. International Journal of Systems Science 32, 1345-1351.
Ku, C. C. (2016). Robust mixed H_2/*Passivity* performance controller design for uncertain drum-boiler system. Journal of Marine Science and Technology 24, 1003-1010.
Ku, C. C. and M. T. Li (2015). Mixed H_2/*Passivity* performance controller design for drum-boiler system. Journal of Marine Engineering and Technology 14, 137-145.
Ku, C. C., P. H. Huang and W. J. Chang (2010). Passive fuzzy controller design for nonlinear systems with multiplicative noises. Journal of the Franklin Institute 347, 732-750.
Li, C. G., H. B. Zhang and X. F. Liao (2005). Passivity and passification of uncertain fuzzy systems. IEE Proceedings-Circuits Devices Systems 152, 649-653.
Li, L. and V. A. Ugrinovskii (2007). On necessary and sufficient conditions for H_∞ output feedback control of Markov jump linear systems. IEEE Trans. on Automatic Control 52, 1287-1292.
Liu, Y., Z. Wang and X. Liu (2008). Robust H_∞ filtering for discrete nonlinear stochastic systems with time-varying delay. Journal of Mathematical Analysis and Applications 341, 318-336.
Lozano, R, B. Brogliato, O. Egeland and B. Maschke (2000). Dissipative Systems Analysis and Control Theory and Applications. Springer: London.
Lu, J. and R. E. Skelton (2002). Mean-square small gain theorem for stochastic control: discrete-time case. IEEE Trans. on Automatic Control 47, 490-494.
Ma, M. and H. Chen (2006). Constrained H_2 control of active suspensions using LMI optimization. Proceeding of the 25th Chinese Control Conference, 702-705.
Mahmoud, M. S. (2000). Robust H_∞ control of linear neutral systems. Automatica 36, 757-764.
Mathiyalagan K., J. H. Park, H. Y. Jung and R. Sakthivel (2015). Non-fragile observer-based H_∞ control for discrete-time systems using passivity theory. Circuits, Systems, and Signal Processing 34, 2499-2516.
Orihuela, L., P. Millán, C. Vivas and F. R. Rubio (2015). H_2/H_∞ control for discrete TDS with application to networked control systems: periodic and asynchronous communication. Optimal Control Applications and Methods 16, 60-76.
Peres, P. L. D. and J. C. Geromel (1993). H_2 control for discrete-time systems optimality and robustness. Automatica 29, 225-228.
Pipeleers, G., B. Demeulenaere, J. Swevers and L. Vandenberghe (2009). Extended LMI characterizations for stability and performance of linear systems. Systems and Control Letters 58, 510-518.
Qiu, J. (2008). Robust mixed H_2/H_∞ filtering design for discrete-time switched polytopic linear systems. IET Control Theory & Applications 2, 420-430.
Scherer, C. W. (2006). LMI relaxations in robust control. European Journal of Control 12, 3-29.
Tan, Z, C. S. Yeng and X. Lihua (2010). Dissipative control of linear discrete-time systems with dissipative uncertainty. International Journal of Control 73, 317-328.
Tanaka, K. and M. Sano (1994). A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer. IEEE Trans. on Fuzzy Systems 2, 119-134.
Wang, H. and Q. Zhu (2015). Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form. Automatica 54, 284-291.
Wang, J., X. M. Zhang and Q. L. Han (2016). Event-triggered generalized dissipativity filtering for neural networks with time-varying delays. IEEE Transactions on Neural Networks and Learning Systems 27, 77-88.
Willmann, G, D. F. Coutinho, L. F. A. Pereira and F. B. Líbano (2007). Multiple-loop H-Infinity control design for uninterruptible power supplies. IEEE Trans. on Industrial Electronics Society 54, 1591-1602.
Xie, S., L. Xie and C. E. D. Souza (1998). Robust dissipative control for linear systems with dissipative uncertainty. International Journal of Control 70, 169-191.
Xu, S., J. Lam and T. Chen (2004). Robust H_∞ control for uncertain discrete stochastic time-delay systems. Systems and Control Letters 51, 203-215.
Yang, G. H., J. S. Yee and J. L. Wang (2002). An iterative LMI method to discrete-time reliable state-feedback controller design with mixed H_2/H_∞ performance. European Journal of Control 8, 126-135.
Zhu, Q. and J. Cao (2012a). Stability analysis of Markovian jump stochastic BAM neural networks with impulse control and mixed time delays. IEEE Transactions on Neural Networks and Learning Systems 23, 467-479.
Zhu, Q. and J. Cao (2012b). Stability of Markovian jump neural networks with impulse control and time varying delays. Nonlinear Analysis Real World Applications 13, 2259-2270.
Zhu, Q. and J. Cao (2015). Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays. Nonlinear Dynamics 79, 1085-1098.
Zhu, Q., F. Jiang, H. Wang and B. Wang (2017). Comment on "Stability analysis of stochastic differential equations with Markovian switching" [Systems & Control Letters 61 (2012) 1209-1214]. Systems & Control Letters 102, 102-103.