MIXED $H_2$/PASSIVITY PERFORMANCES CONTROL OF DISCRETE-TIME LINEAR STOCHASTIC SYSTEMS

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ABSTRACT

A mixed performance control problem of discrete-time linear stochastic systems is discussed and investigated subject to $H_2$ and passivity performances in this paper. Based on Itô modeling approach, stochastic systems can be represented as deterministic difference equation with multiplicative noise term. For the stochastic systems, $H_2$ minimization problem and passivity constraint are simultaneously considered to achieve minimum output energy and attenuation performance. Applying Lyapunov theory, some sufficient conditions are derived into extended Linear Matrix Inequality (LMI) form to apply convex optimization algorithm. Moreover, a mixed $H_2$/Passivity performance controller can be designed such that asymptotical stability and required performances of closed-loop system are guaranteed in the mean square. Finally, some simulations are proposed to demonstrate effectiveness and applicability of the proposed design method.

I. INTRODUCTION

System stability and control performance are two important issues in control engineering. For stability issue, Lyapunov function provides a powerful tool for linear systems and nonlinear systems. On the other hand, various schemes have been employed to achieve the required performance. For example, $H_2$ scheme (Peres and Geromel, 1993; Du, 2006; Ma and Chen, 2006) is applied to minimize a quadratic control performance index. The $H_\infty$ scheme (Mahmoud, 2000; Li and Ugrinovskii, 2007; Willems et al., 2006) proposes performance index to achieve robustness of systems. Moreover, some mixed $H_2/H_\infty$ performance schemes were developed by Kim (2001), Yang et al. (2002), Qiu (2008), Fioravanti et al. (2014) and Orihuela et al. (2015) through combining the merits of optimal $H_2$ and robust $H_\infty$ control schemes. Usually, the purpose of mixed $H_2/H_\infty$ performance scheme is to minimize upper bound of $H_2$ performance under a desired $H_\infty$ norm bound constraint. Thus, the mixed $H_2/H_\infty$ performance scheme is a more attractive design method in engineering practice because sole control performance is a worst case design that leads conservative. In order to extend the application of mixed performance controller design method, its generality and flexibility are an interesting issue.

To propose a general and flexible mixed performance control criterion, the passivity theory (Jiang and Hill, 1998; Xie et al., 1998; Lozano et al., 2000; Tan et al., 2010) is considered for achieving optimal performance in this paper. Referring to (Lozano et al., 2000), Willems developed power supply function expressing passivity theory based on conservation, dissipation and transport of system energy. In general, passivity theory provides useful tool to analyze stability of linear systems and nonlinear systems. Based on the energy concept, the passivity theory was furtherly applied to deal with attenuation performance for fuzzy control (Li et al., 2005; Ku et al., 2010), observer-based control (Mathiyalagan et al., 2015), filter design (Wang et al., 2016) and so on. Through setting power supply function (Lozano et al., 2000; Ku et al., 2010), it is easily found that the passivity theory provides a formulation including $H_\infty$ constraint, positive real theory and passive types. It should be noted that the $H_\infty$ scheme is a special case of passivity theory. According to the above description, the passivity theory is applied to constrain the effect of external disturbance on the system in this paper. Thus, a mixed $H_2$/Passivity performance control criterion is proposed to ensure $H_2$ performance under the desired disturbance attenuation performance. The similar mixed $H_2$/Passivity performance controller design method was developed by Ku and Li (2015) and Ku (2016) for continuous-time deterministic systems. To extend the applicability of Ku and Li (2015) and Ku (2016), mixed performance control problem of the discrete-time linear stochastic system is discussed and solved in this paper.

In practical control, stability analysis and controller synthesis of stochastic systems are always challenging problems according to characteristics as unmeasurable and unpredictable dynamics. Since the development of Itô stochastic modelling approach
In this paper, a discrete-time linear stochastic system is described as follows:

\[ x(k+1) = \mathbf{A}x(k) + \mathbf{B}u(k) + \mathbf{E}v(k) + (\mathbf{A}_e x(k) + \mathbf{B}_e u(k) + \mathbf{E}_e v(k))w(k) \]  

\[ y(k) = \mathbf{C}_x x(k) + \mathbf{D}_x v(k) \]  

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( y(k) \in \mathbb{R}^m \) is the measured output vector, \( z(k) \in \mathbb{R}^r \) is the controlled output vector, \( u(k) \in \mathbb{R}^q \) is the control input vector, \( v(k) \in \mathbb{R}^s \) is the disturbance input vector, and \( w(k) \) is a scalar discrete type Brownian motion. \( \mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{C}_x, \mathbf{D}_x, \mathbf{E}_e, \mathbf{A}_e, \mathbf{B}_e, \mathbf{E}_e \) are constant matrices with compatible dimensions. Referring to (Karatzas and Shreve, 1991), one can find the independent increment properties of \( w(k) \), such as \( E[w(k)] = 0, E[w(k)w(k)] = 0 \) and \( E[w(k)w(k)] = 1 \), where \( E[\cdot] \) denotes the expected value of \( \cdot \). Besides, it should be noted that the pair \( (\mathbf{A}, \mathbf{B}) \) is known and controllable.

For dealing with stabilization problem of (1), the following state feedback controller is considered in this paper.

\[ u(k) = \mathbf{F}x(k) \]  

where \( \mathbf{F} \in \mathbb{R}^{m \times n} \) is a feedback gain and is needed to be designed. Substituting (2) into (1), the following closed-loop system is inferred.

\[ x(k+1) = \mathbf{A}_f x(k) + \mathbf{E}v(k) + (\mathbf{A}_e x(k) + \mathbf{E}_e v(k))w(k) \]  

\[ y(k) = \mathbf{C}_x x(k) + \mathbf{D}_x v(k) \]  

\[ z(k) = \mathbf{C}_z x(k) \]  

where \( \mathbf{A}_f = \mathbf{A} + \mathbf{BF}, \mathbf{A}_e = \mathbf{A} + \mathbf{BF} \) and \( \mathbf{C}_z = \mathbf{C}_x + \mathbf{D}_x \mathbf{F} \).

In this paper, the passivity theory is substituted for the \( H_\infty \) scheme to constrain the effect of external disturbance on the closed-loop system (3). Referring to Lozano et al. (2000), the passivity theory can be introduced in the following definition.

**Definition 1:** The closed-loop system (3) with disturbance input \( v(k) \) and measured output \( y(k) \) is called passive if there exists matrices \( \mathbf{S}_1, \mathbf{S}_2 \geq 0 \) and \( \mathbf{S}_3 \) such that

\[ E \left\{ \sum_{k=0}^{k_2} y^T(k) \mathbf{S}_1 y(k) \right\} \geq E \left\{ \sum_{k=0}^{k_2} y^T(k) \mathbf{S}_2 y(k) + \sum_{k=0}^{k_2} v^T(k) \mathbf{S}_3 v(k) \right\} \]  

for any terminal time \( k_2 > 0 \).

From (Ku et al., 2010), power supply function (4) can be reduced into several cases for dealing with attenuation perfor-
mance of the closed-loop system (3). For example, inequality (4) is reduced as a) $H_\alpha$ performance by setting $S_1 = 0$, $S_2 = I$ and $S_3 = -\gamma^2$ with positive scalar $\gamma$; b) positive real performance by setting $S_1 = I$, $S_2 = 0$ and $S_3 = 0$; c) strictly input passive performance by setting $S_1 = I$, $S_2 = 0$ and $S_3 = \gamma I$ with positive scalar $\gamma$; d) strictly output passive performance by setting $S_1 = I$, $S_2 = \gamma I$ and $S_3 = 0$ with positive scalar $\gamma$; e) strictly vary passive performance by setting $S_1 = I$, $S_2 = \gamma I$ and $S_3 = \gamma I$ with positive scalars $\gamma$ and $\zeta$. Thus, the passivity theory in Definition 1 proposes a general and flexible attenuation performance index. Besides, in case of $v(k) = 0$, the $H_2$ performance index defined in the following definition is applied to minimize output energy.

**Definition 2 (Kim, 2001):** The controller (2) is an $H_2$ performance measure for the closed-loop system (3) with $v(k) = 0$, if one can find a $\alpha > 0$ to satisfy the following inequality.

$$
E\left\{\sum_{k=0}^{T_f} z^T(k) z(k)\right\} < \alpha
$$

(5)

where $T_f > 0$ is terminal time of control.

**Remark 1**

Based on the definitions of this paper, the mixed performance of the closed-loop system (3) is dealt with achieving $H_2$ performance and passivity. With setting $S_1 = 0$, $S_2 = I$ and $S_3 = -\gamma^2$, the proposed mixed $H_2$/Passivity performance criterion can be reduced as the mixed $H_2/H_\alpha$ performance criterion. In addition, the control issue discussed in this paper is more general than the design method of Fioravanti et al. (2014) according to the consideration of stochastic behavior. Concluding the above description, the proposed design method is more general and flexible than the existing design methods (Kim, 2001; Yang et al., 2002; Fioravanti et al., 2014; Orihuela et al., 2015).

Through Definition 1 and Definition 2, two goals are concerned to guarantee the required control performance. One of the goals is to ensure the attenuation performance of the closed-loop system (3). Another goal, in case of $v(k) = 0$, is to minimize output energy of the closed-loop system (3) with initial conditions. According to the above illustrations, the mixed $H_2$/Passivity performance control problem of the closed-loop system (3) is discussed in the next section.

### III. MIXED $H_2$/PASSIVITY PERFORMANCE CONTROLLER DESIGN METHOD

In this section, some sufficient conditions are derived by Lyapunov function to achieve the above definitions. Through solving the sufficient conditions, some feasible solutions can be obtained to establish mixed $H_2$/Passivity performance controller (2) to stabilize closed-loop system (3).

**Theorem 1**

Given matrices $S_1, S_2 \geq 0$ and $S_3$, if there exists a minimized positive scalar $\alpha$, positive definite matrix $P$ and feedback gain $F$ satisfying the following inequalities then the asymptotical stability and mixed $H_2$/Passivity performance of the closed-loop system (3) are achieved in the mean square.

$$
\begin{bmatrix}
A_f^T P A_f + \bar{A}_f^T \bar{P} A_f - P + C_f^T S_2 C_f \\
E_f^T P A_f + \bar{E}_f^T \bar{P} A_f - S_1^T C_1 + D_f^T S_2 C_1 \\
* & E_f^T P E_f + \bar{E}_f^T \bar{P} E_f + S_1 - D_f^T S_1 - S_1^T D_f + D_f^T S_2 D_f
\end{bmatrix} < 0
$$

(6)

$$
A_f^T P A_f + \bar{X}_f^T \bar{P} A_f - P + C_{z_f}^T C_{z_f} < 0
$$

(7)

$$
x^T(0) P x(0) - \alpha < 0
$$

(8)

where $*$ denotes the transposed elements or matrices for symmetric position.

**Proof:**

Let us choose the following Lyapunov function.

$$
V(x(k)) = x^T(k) P x(k)
$$

(9)

Taking the first forward difference of (9), one has

$$
\Delta V(x(k)) = x^T(k+1) P x(k+1) - x^T(k) P x(k)
$$

$$
= \left( A_f x(k) + E_f v(k) + (\bar{X}_f x(k) + \bar{E}_f v(k)) w(k) \right)^T P
$$

$$
\times \left( A_f x(k) + E_f v(k) + (\bar{X}_f x(k) + \bar{E}_f v(k)) w(k) \right)
$$

$$
- x^T(k) P x(k)
$$

(10)

Taking expectation of (10), the following equation can be obtained with the independent increment property of Brownian motion (Karatzas and Shreve, 1991).

$$
E\left\{\Delta V(x(k))\right\}
$$

$$
= E \left\{ x^T(k+1) P x(k+1) - x^T(k) P x(k) \right\}
$$

$$
= E \left[ \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}^T \begin{bmatrix} A_f^T P A_f + \bar{X}_f^T \bar{P} A_f - P & * \\ E_f^T P A_f + \bar{E}_f^T \bar{P} A_f & E_f^T P E_f + \bar{E}_f^T \bar{P} E_f \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \right]
$$

(11)
Let us define the following cost function with zero initial condition.

$$\Gamma(x, v, k) = E \left\{ \sum_{k=0}^{\infty} \left[ y^T(k) S_2 y(k) + v^T(k) S_1 v(k) \right] \right\}$$

$$= E \left\{ \sum_{k=0}^{\infty} \left[ y^T(k) S_2 y(k) + v^T(k) S_1 v(k) \right] \right\}$$

$$+ 2 \sum_{k=0}^{\infty} y^T(k) S_1 v(k)$$

$$+ E \left\{ 2 \sum_{k=0}^{\infty} y^T(k) S_1 v(k) \right\}$$

Because (16) is equivalent to (4), the closed-loop system (3) is passive with the given $S_1, S_2 \geq 0$ and $S_3$. Next, the asymptotical stability of the closed-loop system (3) with zero disturbance input is analyzed in the following derivative. When the external disturbance is zero, i.e., $v(k) = 0$, Eq. (11) can be rewritten as the following relations.

$$E \left\{ \Delta V(x(k)) \right\}$$

$$= E \left\{ x^T(k) \left( A^T P A + \bar{A}^T \bar{P} \bar{A} - P \right) x(k) \right\}$$

Introducing (3c) into (17), one has

$$E \left\{ \Delta V(x(k)) \right\}$$

$$= E \left\{ x^T(k) \left( A^T P A + \bar{A}^T \bar{P} \bar{A} - P \right) x(k) \right\}$$

If inequality (7) holds, then one can find $E\{\Delta V(x(k))\} < 0$ from (18). Referring to (Ghaoui, 1995) and $E\{\Delta V(x(k))\} < 0$, the closed-loop system (3) with zero external disturbance input is asymptotically stable in the mean square. Besides, condition (7) implies the following inequality.

$$E \left\{ \Delta V(x(k)) + z^T(k) z(k) \right\} < 0$$

and

$$E \left\{ \Delta V(x(k)) \right\} < E \left\{ -z^T(k) z(k) \right\}$$

Summarizing (20) from 0 to $T_f$, we have

$$E \left\{ x^T \left( T_f \right) P x \left( T_f \right) - x^T(0) P x(0) \right\} < E \left\{ \sum_{k=0}^{T_f} z^T(k) z(k) \right\}.$$
system (3) is asymptotically stable, that implies \( x(T_r) \to 0 \) and \( x^T(T_r)P_x(T_r) \to 0 \) as \( T_r \to \infty \). Therefore, inequality (21) can be further inferred as the following inequality.

\[
E \left\{ \sum_{k=0}^{T_r} z^T(k) z(k) \right\} \leq E \left\{ x^T(0) P x(0) \right\} < 0
\]  

(22)

From inequality (22), it is known that \( E \{ x^T(0) P x(0) \} \) is the upper bound of output energy. If condition (8) holds, then the following inequalities can be obtained due to \( E \{ a \} = a \) where \( a \) is scalar.

\[
E \{ x^T(0) P x(0) - \alpha \} \leq 0
\]  

(23)

and

\[
E \{ x^T(0) P x(0) \} \leq \alpha
\]  

(24)

From (22) and (24), the following relation can be directly found.

\[
E \left\{ \sum_{k=0}^{T_r} z^T(k) z(k) \right\} < \alpha
\]  

(25)

Because (25) is equivalent to (5), one knows that if conditions (7) and (8) of this theorem are satisfied, then the \( H_2 \) performance of the closed-loop system (3) is achieved. Moreover, the output energy is minimized subject to \( \alpha \). The proof of this theorem is completed.

In Theorem 1, the sufficient conditions are derived by using Lyapunov function to discuss the asymptotical stability and mixed \( H_2/\text{Passivity} \) performance of (3) in the mean square. To apply convex optimal algorithm (Boyd and Vandenberghe, 2004), the sufficient conditions in Theorem 1 are converted into LMI form in the following theorem.

**Theorem 2**

For the given matrices \( S_1, S_2 \geq 0 \) and \( S_0 \), the asymptotical stability and mixed \( H_2/\text{Passivity} \) performance of closed-loop system (3) are achieved in the mean square if there exists positive scalar \( \alpha \), positive definite matrix \( P \), arbitrary matrix \( G \) and feedback gain \( F \) such that

\[
\begin{bmatrix}
X - G^T - G & * & * & * \\
-S_1 C G & S_1 - D_1^T S_1 - S_0 D_1 & * & * \\
AG + BK & E & -X & * \\
AG + \bar{B}K & \bar{E} & 0 & -X \\
C G & D_1 & 0 & 0 -S_2^T \\
\end{bmatrix} < 0
\]  

(26)

where \( X = P^{-1} \) and \( K = FG \).

**Proof:**

Applying Schur complement (Boyd et al., 1994), inequality (6) can be converted into the following inequality.

\[
\begin{bmatrix}
-P & * & * & * & * \\
-S_1 C G & S_1 - D_1^T S_1 - S_0 D_1 & * & * & * \\
A C G & E & -P^{-1} & * & * \\
\bar{A} C G & \bar{E} & 0 & -P^{-1} & * \\
C G & D_1 & 0 & 0 & -S_2^T \\
\end{bmatrix} < 0
\]  

(29)

Multiplying the both side of (29) by \( \text{diag}\{G^T, I, I, I, I\} \) and \( \text{diag}\{G, I, I, I, I\} \), where \( \text{diag}\{\cdot\} \) denotes the diagonal matrix with element \( \cdot \), one has

\[
\begin{bmatrix}
-G^T P^{-1} G & * & * & * & * \\
-S_1 C G & S_1 - D_1^T S_1 - S_0 D_1 & * & * & * \\
A C G & E & -P^{-1} & * & * \\
\bar{A} C G & \bar{E} & 0 & -P^{-1} & * \\
C G & D_1 & 0 & 0 & -S_2^T \\
\end{bmatrix} < 0
\]  

(30)

According to \( P > 0 \), one holds the following fact.

\[
(P^{-1} - G^T P^{-1} G) \geq 0
\]  

(31)

Arranging (31) one can find the following relation to replacing the bilinear term in (30).

\[
P^{-1} G^T - G \geq -G^T P G
\]  

(32)

Thus, one has

\[
\begin{bmatrix}
X - G^T - G & * & * & * & * \\
-S_1 C G & S_1 - D_1^T S_1 - S_0 D_1 & * & * & * \\
A C G & E & -P^{-1} & * & * \\
\bar{A} C G & \bar{E} & 0 & -P^{-1} & * \\
C G & D_1 & 0 & 0 & -S_2^T \\
\end{bmatrix} < 0
\]  

(33)
Substituting $A_f = A + BF$ and $\bar{A}_f = \bar{A} + \bar{B}F$ into (33), the following relation can be obtained by setting $X = P^T$ and $K = FG$.

$$
\begin{bmatrix}
X - G^T G & * & * & * \\
-S_i^T C_i G & S_3 - D_3 S_1 - S_i D_i & * & * \\
AG + BK & E & -X & * \\
\bar{X}G + \bar{B}K & \bar{E} & 0 & -X \\
C_i G & D_i & 0 & 0 -S_i^{-1} \\
\end{bmatrix} < 0 \quad (34)
$$

It is easy to find that (34) is equivalent to (26). Thus, if condition (26) holds, then condition (6) is satisfied. Besides, (27) and (28) can be respectively derived from conditions (7) and (8) with the above converting processes. Thus, the proofs of (28) and (29) are omitted here. The proof of Theorem 2 is complete.

In Theorem 2, the sufficient conditions are converted into extended LMI form via introducing arbitrary matrix and applying transformation technologies. Therefore, the feasible solutions can be obtained via using convex optimization algorithm to establish controller (2) such that the $H_2$ performance and passivity of the closed-loop system (3) are achieved.

**Remark 2**

Referring to (Pipeleers et al., 2009), one can find that the extended LMI form possesses two advantages. One of the advantages is that the extended LMI form reduces the conservatism in finding feasible solutions for the conditions in Theorem 2. Another advantage is that the sufficient conditions in Theorem 2 can be reduced to standard LMI form by setting $G = P^T$.

By setting $S_2 = 0$, a sufficient condition (26) in Theorem 2 becomes as non-standard LMI form that is difficult to be solved by the convex optimal algorithm. For the case as $S_2 = 0$, the sufficient conditions in Theorem 2 can be rewritten as the following corollary.

**Corollary 1**

For the given matrices $S_1$ and $S_2$, the asymptotical stability and mixed $H_2$/Passivity performance of closed-loop system (3) are achieved in the mean square, if there exists positive scalar $\alpha$, positive definite matrix $P$, arbitrary matrix $G$ and feedback gain $F$ such that

$$
\begin{bmatrix}
X - G^T G & * & * & * \\
-S_i^T C_i G & S_3 - D_3 S_1 - S_i D_i & * & * \\
AG + BK & E & -X & * \\
\bar{X}G + \bar{B}K & \bar{E} & 0 & -X \\
C_i G & D_i & 0 & 0 -S_i^{-1} \\
\end{bmatrix} < 0 \quad (35)
$$

$$
\begin{bmatrix}
X - G^T G & * & * & * \\
C_i G + D_i K & -I & * & * \\
AG + BK & 0 & -X & * \\
\bar{X}G + \bar{B}K & 0 & 0 & -X \\
\end{bmatrix} < 0 \quad (36)
$$

Proof:

Following the proof procedure of Theorem 2 and setting $S_2 = 0$, the proof of this theorem can be easily obtained and it is omitted here.

Based on Corollary 1, one can also apply convex optimal algorithm to find feasible solutions to establish controller (2) for the closed-loop system (3). For demonstrating effectiveness and usefulness of the proposed design method, some numerical simulations are proposed in the following section.

**IV. SIMULATION**

In this section, two numerical examples are proposed. In the first example, a dissipative controller design method (Tan et al., 2010) is applied to compare with the proposed design method. Through the simulation results in the first example, one can find that the considerations of stochastic behavior and $H_2$ performance are important issues in practical control problem. In the second example, a mixed performance control problem of an inverted pendulum on a cart system is discussed and solved by the proposed design method. Moreover, a comparison between the proposed design method and a mixed $H_2$/Passive performance design method (Fioravanti et al., 2014) is provided in Example 2. Through this example, one can find that the proposed design method can be reduced into the design method (Fioravanti et al., 2014). And, the proposed design method possesses less conservatism than the method (Fioravanti et al., 2014) based on the simulation results.

**Example 1**

Referring to (Tanaka and Sano, 1994), the nonlinear dynamic equations of truck-trailer system can be proposed as follows:

$$
x_1(k+1) = 1.363x_1(k) - 0.7143u(k) + 0.1v(k) + 0.09x_1(k) - 0.45u(k) - 0.01v(k)w(k)
$$

$$
x_2(k+1) = -0.363x_1(k) + x_2(k) + 0.1x_2(k)w(k)
$$

$$
x_3(k+1) = -2x_1(k) + x_3(k) + x_2(k)
$$

$$
y(k) = x_1(k) + v(k)
$$

where $x_1(k)$ is the angle difference truck and trailer, $x_2(k)$ is the angle of the trailer, and $x_3(k)$ is the vertical position of the rear
The Proposed Design Method
Method of Tan et al., 2010

FIG. 1. Responses of state $x_1(k)$ in Example 1.

Fig. 1. Responses of state $x_1(k)$ in Example 1.

Fig. 2. Responses of state $x_2(k)$ in Example 1.

Fig. 3. Responses of state $x_3(k)$ in Example 1.

end of the trailer. Moreover, the external disturbance $v(k)$ is zero-mean white noise with variance 0.1. Besides, the following linear model can be obtained to represent the local behavior of (38) around the equilibrium point $x^*(k) = [0 0 0]^T$. Moreover, the output $z(k)$ is added to achieve $H_2$ performance.

$$x(k+1) = Ax(k) + Bu(k) + Ev(k) + \tilde{A}x(k)w(k)$$  \hspace{1cm} (39a)

$$y(k) = C_1x(k)$$  \hspace{1cm} (39b)

$$z(k) = C_2x(k) + D_2u(k)$$  \hspace{1cm} (39c)

where $A = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1 & 0 \\ 0.3636 & 2 & 1 \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} 0.09 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$, $B = \begin{bmatrix} -0.7143 \\ 0 \\ 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $D_1 = D_2 = 1$ and $E = [0.01 0 0]^T$. In order to apply the proposed design method, the $S_1 = 1$, $S_2 = 0.8$ and $S_3 = 0.8$, initial condition $x(0) = \begin{bmatrix} -40^- \\ -20^- \\ 1 \end{bmatrix}^T$ and sampled period as 2 second are determined. And then, the following controller can be established via solving the sufficient conditions in Theorem 2.

$$u(k) = Fx(k)$$  \hspace{1cm} (40)

where $F = \begin{bmatrix} 2.6878 & -3.3552 & 0.5516 \end{bmatrix}$. Moreover, the minimum value of $\alpha = 4.9961$ is also obtained. Applying (40), the responses of (38) are stated in Figs. 1-3 with the initial condition. Besides, the achievement of performances can be checked by the following equations with the simulation results.

$$\frac{1}{L_k} \sum_{k=0}^{L_k} y^T(k)S_1v(k) = 1.006$$  \hspace{1cm} (41)$$

and

$$\sum_{k=0}^{L_k} y^T(k)S_2y(k) + \sum_{k=0}^{L_k} v^T(k)S_3v(k)$$

According to the above equations, the value of (41) is bigger than one that satisfies Definition 1. Moreover, the value in (42) is smaller than the obtained minimum $\alpha$ that achieves Definition 2. From (41), (42) and Figs. 1-3, the asymptotical stability and mixed $H_2$ / Passivity performance of truck-trailer system (38) are thus achieved by the controller (40).
Referring to (Tan et al., 2010), a dissipative controller design method was proposed without considering $H_2$ performance and stochastic behaviors. Based on the same matrices $S_1 = I$, $S_2 = 0.8$ and $S_3 = 0.8$, the following controller can be designed by using the method (Tan et al., 2010).

$$u(k) = Fx(k)$$  \hfill (43)

where $F = \begin{bmatrix} 2.2473 & -1.1462 & 0.0806 \end{bmatrix}$. With the same initial condition, the responses of (38) driven by (43) are also proposed in Figs. 1-3. Based on those responses, the following values can be obtained.

$$2\sum_{k=0}^{k} y^T(k)S_1 v(k)$$

$$\sum_{k=0}^{k} y^T(k)S_2 y(k) + \sum_{k=0}^{k} v^T(k)S_1 v(k)$$

and

$$\sum_{k=0}^{k} z^T(k)z(k) = 47$$ \hfill (45)

Because the value in (44) is bigger than one, the passivity performance of (38) driven by (43) is achieved. However, the value in (45) is bigger than the obtained $\alpha = 4.9961$. Thus, the $H_2$ performance of (38) driven by (43) is not achieved. It means that the output energy of (38) driven by (43) is bigger than that driven by (40).

Referring to Figs. 1-3, controller (40) possesses better settling time than controller (43). Besides, the overshoot of (38) driven by (43) is bigger than that driven by (40). The poor control performance of (43) is caused by the considered stochastic behavior. Moreover, because the $H_2$ performance was not concerned by Tan et al. (2010), the response of $x_3(k)$ cannot be constrained under the given requirement. Concluding this simulation results, the proposed design method provides better control performance than the method of Tan et al. (2010).

**Example 2**

In this example, a comparison between the proposed design method and the method (Fioravanti et al., 2014) is provided to show advantages of this paper. From (Fioravanti et al., 2014), a mixed $H_2/H_\infty$ performance controller design method has been proposed for discrete-time linear systems. In this example, two cases are proposed to emphasize the contribution of this paper. One of the cases is to show that the proposed design method provides less conservative than the method (Fioravanti et al., 2014). Another case is to emphasize the importance of considering stochastic behavior under the same performance indexes. Referring to (Iordanou and Surgenor, 1997), the following discrete-time inverted pendulum on a cart system was modeled with sampling time as 0.01 second. To apply the proposed design method, an external disturbance $v(k)$ and a multiplicative noise $w(k)$ are added.

$$x(k+1) = A x(k) + B u(k) + E v(k)$$

$$+ (\bar{A} x(k) + \bar{B} u(k) + \bar{E} v(k)) w(k)$$ \hfill (46a)

$$y(k) = C_1 x(k) + D_1 v(k)$$ \hfill (46b)

$$z(k) = C_2 x(k) + D_2 u(k)$$ \hfill (46c)

where $x(k) = [x_1(k) \ x_2(k) \ x_3(k) \ x_4(k)]^T$, $x_1(k)$ is the cart position, $x_2(k)$ is the cart velocity, $x_3(k)$ is the payload angle, $x_4(k)$ is the payload angle velocity, $u(k)$ is the applied force, and $v(k)$ is the zero-mean white noise with unit variance.

And, system matrices are presented by $E = [0 \ 0 \ 0.01 \ 0]^T$, $C_1 = [0 \ 0 \ 1 \ 0]^T$, $C_2 = [1 \ 0 \ 0 \ 0]^T$, $D_1 = D_2 = 1$ ,

$$A = \begin{bmatrix} 1 & 0.0087 & 0 & 0 \\ 0 & 0.7515 & 0 & 0 \\ 0 & -0.0111 & 1.0015 & 0.0100 \\ 0 & -2.1235 & 0.3052 & 0.9999 \end{bmatrix}$$ \hfill and $B = \begin{bmatrix} 0.0027 \\ 0.5219 \\ 0.0234 \\ 4.4593 \end{bmatrix}$

In order to compare with the method (Fioravanti et al., 2014), the matrices in Definition 1 are set as $S_1 = 0$, $S_2 = I$ and $S_3 = \gamma^2$ such that the proposed design method is reduced as mixed $H_2/H_\infty$ performance controller design method. And, the initial condition $x(0) = [0.5 \ 0 \ 50^0 \ 0]^T$ is assumed in the following cases.

**Case 1**

In the first case, the multiplicative noise terms in (46a) are set as zero ($\bar{A} = \bar{B} = \bar{E} = 0$). Moreover, the attenuation level $\gamma$ and upper bounded $\alpha$ are respectively fixed to apply the proposed design method and the method (Fioravanti et al., 2014) to solve the mixed performance control problem of (46). Through MATLAB LMI Toolbox, simulation results of this case are concluded in Table 1. From Table 1, by fixing $\gamma$, the value as $\alpha = 30.4778$ can be obtained by the proposed design method. And, the value of $\alpha = 32.0344$ is obtained by applying the method (Fioravanti et al., 2014). It is easy to see that the smaller upper bound $\alpha$ can be found by using the proposed design method than one found by Fioravanti et al. (2014). Besides, under fixing $\gamma$, the attenuation level $\gamma = 1.2093$ is obtained by the proposed design method. Moreover, an attenuation level $\gamma = 1.8334$ is obtained by the method (Fioravanti et al., 2014). It should be noted that the proposed design method provides smaller attenuation level than the method (Fioravanti et al., 2014). It means that the proposed design method provides better attenuation performance than the method (Fioravanti et al., 2014).
Table 1. Comparison Between the Proposed Design Method and Method of (Fioravanti et al., 2014).

<table>
<thead>
<tr>
<th>Method</th>
<th>Fixing $\gamma = 1.1$</th>
<th>Fixing $\alpha = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Proposed Design Method</td>
<td>$\alpha = 30.4778$</td>
<td>$\gamma = 1.2093$</td>
</tr>
<tr>
<td>Method of (Fioravanti et al., 2014)</td>
<td>$\alpha = 32.0344$</td>
<td>$\gamma = 1.8334$</td>
</tr>
</tbody>
</table>

et al., 2014). Based on those simulation results, one can find that the conservatism of the proposed design method is less than one of the method (Fioravanti et al., 2014) according to extended LMI form. Therefore, the proposed design method is less conservative than the method (Fioravanti et al., 2014).

Case 2

In this case, the matrices of multiplicative noise term of (46) are set as

$$\bar{A} = \begin{bmatrix} 0.07 & 0.01 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0.001 & 0.3 & 0 \\ 0 & 0.1 & 0 & 0.01 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.05 \\ 0.03 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 0.001 \\ 0.002 \\ 0.003 \end{bmatrix}.$$

Moreover, minimum upper bound $\alpha = 20$ and attenuation level $\gamma = 2$ are simultaneously fixed. Applying the proposed design method, the following controller can be established.

$$u(k) = Fx(k)$$

(47)

where $F = \begin{bmatrix} 1.0165 & 1.4476 & -1.5815 & -0.3116 \end{bmatrix}$. Applying (47), the responses of (46) are stated in Figs. 4-7. Based on those responses, the following values can be obtained to check achievements of Definition 1 and Definition 2, respectively.
Referring to the above simulation results, asymptotical stability and mixed $H_2/H_\infty$ performance of (46) can be achieved by using (47) or (50). However, from Figs. 4-7, one can easily find that the overshoot of (46) driven by (47) is smaller than that driven by (50). Moreover, the settling time of (46) driven by (47) is short than that driven by (50). Those poor performances in controller (50) designed by Fioravanti et al. (2014) are caused via the stochastic behaviors. According to the simulation results, it can be thus concluded that the proposed mixed $H_2$/Passivity controller design method proposes some improvements for (Fioravanti et al., 2014) in stabilizing the discrete-time linear stochastic systems.

Concluding the simulation results of this section, the consideration of stochastic behavior is an important issue in practical control because it always causes poor performance during control process. Besides, the derived sufficient conditions in Theorem 2 provide some relaxations in searching feasible solutions according to an arbitrary matrix $G$. Therefore, the proposed design method is not only less conservative but also more general than the methods (Tan et al., 2010; Fioravanti et al., 2014).

V. CONCLUSIONS

In this paper, a general and flexible mixed performance control problem of the discrete-time linear stochastic systems has been discussed by $H_\infty$ control scheme and passivity theory to achieve minimized output energy and external disturbance constraint. To proposed mixed $H_2$/Passivity performance controller design method, some sufficient conditions were derived by Lyapunov function. Moreover, the derived sufficient conditions were converted into extended LMI form to reduce some conservatism in searching feasible solutions. Therefore, one can find the feasible solutions to build a controller such that the asymptotical stability and mixed $H_2$/Passivity performance of the stochastic system are achieved in the mean square. Finally, some simulation results have been proposed to show the effectiveness and applicability of the proposed design method.

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