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SUPER-EFFICIENT OR SUPER-INEFFICIENT? INSIGHTS FROM A JOINT COMPUTATION MODEL FOR SLACKS-BASED MEASURE IN DEA: A NOTE

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ABSTRACT

Chen (2013) extended the exiting SBM approaches and developed a joint model (J-SBM) that can (1) simultaneously compute SBM scores for inefficient DMUs and super-efficiency for efficient DMUs, (2) guarantee the reference points generated by the J-SBM are Pareto-efficient, and (3) the J-SBM scores of a DMU are continuous in the input and output space. In this note, we provide a counter example which shows that the reference point generated by the J-SBM might not be Pareto-efficient, and the current study has also revised the model (8) proposed by Chen (2013).

I. INTRODUCTION

Data envelopment analysis (DEA) is first proposed by Charnes et al. (1978) as a non-parametric approach to measure the relative efficiency of a decision making unit (DMU). Since then many papers have been published on its methodology and applications. There are two types of DEA models, the radial and non-radial models. The CCR model measures the radial efficiency of the inputs (input-oriented) or outputs (output-oriented) by gauging the ratio of the inputs to be contracted or the ratio of the outputs to be enlarged so that the evaluated DMU is moved to the efficient frontier. One of the shortcomings in radial efficiency is that it could not account for all inefficiency of a DMU (Morita et al., 2005). Slacks need to be considered simultaneously with radial efficiency to identify the Pareto-efficient projection of a DMU. To overcome this, Charnes et al. (1985) developed additive model of DEA, which deals with input excesses and output shortfalls directly. However, the additive DEA model does not provide an efficiency score between zero and one. Tone (2001) developed a slacks-based measure (SBM) of efficiency, which directly deals with input and output slacks in data envelopment analysis and provides an efficiency score between zero and one, as in the radial DEA model. Tone (2002) further presented a super-efficiency model (S-SBM) for decision making units that are efficient under SBM so that efficient DMUs can be discriminated (or ranked) as opposed to the radial super-efficiency measure proposed by Andersen and Petersen (1993).

Chen (2013) highlighted three major implementation issues for the S-SBM model. The first is that S-SBM model can only be used to compute super-efficiency scores for efficient DMUs, but not SBM scores for inefficient DMUs. The second is that S-SBM may identify a weakly efficient reference point. The third is that discontinuous gap exists between the SBM and S-SBM scores of a weakly efficient DMU when it is subject to small perturbations of input-output data. Chen (2013) proposes a new model (J-SBM) to simultaneously tackle these issues. Later in the corrigendum (Chen, 2014), Chen pointed out that the J-SBM model may not be continuous in the input-output space when the reference point changes to another frontier point under the variation of inputs and/or outputs. In this note, we further point out that J-SBM model does not resolve the second issue. The reference point identified by J-SBM model may still be weakly efficient. In addition, the proposed model (8) in Chen (2013) might have a negative objection function value and becomes unbounded, which is also resolved in this note. This note is organized as follows. Section 2 briefly reviews the J-SBM model, and presents a counter example to demonstrate that J-SBM might still identify a weakly efficient reference point. In Section 3, concluding remarks and revision of the proposed model (8) in Chen (2013) are made.

II. COUNTER EXAMPLE

Suppose there are n DMUs associated with m inputs and s outputs. Let x_{ji} denote the ith input of DMUj and y_{jr} denote rth

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output of DMUj. Assume that all data are positive, i.e., x_{ji}, $y_{jr} > 0$ for all possible $i = 1, \ldots, m$; $r = 1, \ldots, s$; $j = 1, \ldots, n$. The SBM model developed by Tone (2001) is as follows:

$$\min \quad \rho_k = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{ki}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+ / y_{kr}}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ji} = x_{ki} - s_i^-, i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{jr} = y_{kr} + s_r^+, r = 1, \ldots, s \\ & \lambda_j \geq 0, j = 1, \ldots, n \\ & s_i^- \geq 0, i = 1, \ldots, m \\ & s_r^+ \geq 0, r = 1, \ldots, s \end{aligned} \tag{1}$$

The reference point identified by (1) is $(x_{ki} - s_i^{-*}, y_{kr} + s_r^{+*})$.

For super-efficiency, the S-SBM model proposed by Tone (2002) is as follows:

$$\min \quad \rho_k^{ssbm} = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{ki}}{1 + \frac{1}{s}\sum_{r=1}^{s} \bar{y}_r / y_{kr}}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ji} \leq \bar{x}_i, i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{jr} \geq \bar{y}_r, r = 1, \ldots, s \\ & \lambda_j \geq 0, j = 1, \ldots, n \\ & x_{ki} \leq \bar{x}_i, i = 1, \ldots, m \\ & y_{kr} \geq \bar{y}_r, r = 1, \ldots, s \\ & \bar{y}_r \geq 0, r = 1, \ldots, s \end{aligned} \tag{2}$$

The reference point identified by (2) is $(\bar{x}_i^*, \bar{y}_r^*)$

Chen (2013) developed the following joint SBM model (J-SBM model) so that SBM and S-SBM can be solved simultaneously and the issues raised by S-SBM can be resolved:

$$\min \phi_k = \frac{JSBM_k^x}{JSBM_k^y} - M(b_1 + (1 - b_1)b_2)$$

$$s.t.\ JSBM_k^x = 1 - \frac{1}{m}[b_1(\sum_{i=1}^{m} \frac{s_i^-}{x_{ki}}) - (1 - b_1)b_2(\sum_{i=1}^{m} \frac{s_i^-}{x_{ki}}) + (1 - b_1)(1 - b_2)(\sum_{i=1}^{m} \frac{\tilde{s}_i^-}{x_{ki}})]$$

$$JSBM_k^y = 1 + \frac{1}{s}[b_1(\sum_{r=1}^{s} \frac{s_r^+}{y_{kr}}) - (1 - b_1)b_2(\sum_{r=1}^{s} \frac{s_r^+}{y_{kr}}) + (1 - b_1)(1 - b_2)(\sum_{r=1}^{s} \frac{\tilde{s}_r^+}{y_{kr}})]$$

$$\text{(I)}: \begin{cases} b_1(\sum_{j=1, j \neq k}^{n} \lambda_j x_{ji}) = b_1(x_{ki} - s_i^-) \forall i \\ b_1(\sum_{j=1, j \neq k}^{n} \lambda_j y_{jr}) = b_1(y_{kr} + s_r^+) \forall r \end{cases}$$

$$\text{(II)}: \begin{cases} (1 - b_1)b_2(\sum_{j=1, j \neq k}^{n} \lambda_j x_{ji}) = (1 - b_1)b_2(x_{ki} + s_i^-) \forall i \\ (1 - b_1)b_2(\sum_{j=1, j \neq k}^{n} \lambda_j y_{jr}) = (1 - b_1)b_2(y_{kr} - s_r^+) \forall r \end{cases}$$

$$\text{(III)}: \begin{cases} (1 - b_1)(1 - b_2)(\sum_{j=1, j \neq k}^{n} \lambda_j x_{ji}) = (1 - b_1)(1 - b_2)(x_{ki} - \tilde{s}_i^-) \forall i \\ (1 - b_1)(1 - b_2)(\sum_{j=1, j \neq k}^{n} \lambda_j y_{jr}) = (1 - b_1)(1 - b_2)(y_{kr} + \tilde{s}_r^+) \forall r \end{cases}$$

$$\begin{aligned} & \lambda_j \geq 0 \text{ for } j = 1, \ldots, n \\ & s_i^- \geq 0, \tilde{s}_i^- \text{ free for } i = 1, \ldots, m \\ & s_r^+ \geq 0, \tilde{s}_r^+ \text{ free for } r = 1, \ldots, s \\ & b_1, b_2 \in \{0, 1\} \\ & M \text{ is a large enough positive number} \end{aligned} \tag{3}$$

The objective function of the J-SBM model is designed to make sure that the constraint set (I) and the constraint set (II) are met first. If there is no feasible solution for these two constraint sets, the constraints switch to constraint set (III), which relaxes the slack variables to free variables. If the target DMU is inefficient, the constraint set (I) will be met. The J-SBM model becomes the SBM model. If the target DMU is efficient, the J-SBM model tries to meet the constraint set (II). If the constraint set (II) is met, the J-SBM model works as S-SBM model. Otherwise, switch to the constraint set (III). Note that the constraints of the S-SBM model can be rewritten as follows if we substitute slacks into the reference point, i.e., let $\bar{x}_i = x_{ki} + s_i^-$ and $\bar{y}_r = y_{kr} - s_r^+$:

$$\begin{aligned} & \sum_{j=1}^{n} \lambda_j x_{ji} \leq x_{ki} + s_i^-, i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{jr} \geq y_{kr} - s_r^+, r = 1, \ldots, s \end{aligned} \tag{4}$$

The difference between (II) and (4) is that the inequalities in (4) become equalities in (II). The logic behind J-SBM model is that the S-SBM model is relaxed if the equalities do not hold. Chen (2013) thought that the equalities will not hold if the reference point is weakly efficient. Under this assumption, the constraint set (II) will not be met if the reference point is weakly efficient, and the constraint set (III) will be used to find a new reference point which is Pareto efficient. Take DMU E in Table 1 as an example, which is an exemplary data set taken from Tone (2002). Chen (2013) revisited the example with J-SBM model. The reference point of E identified by SBM model is (4, 4, 1) which is not Pareto efficient. The reference point will not meet the equalities for the inputs in (II). Therefore, the J-SBM model

Table 1. A data set adopted from Tone (2002).

DMU	x_1	x_2	y
A	4	3	1
B	7	3	1
C	8	1	1
D	4	2	1
E	2	4	1
F	10	1	1
G	12	1	1

Table 2. A counter example data set.

DMU	x_1	x_2	y
A	4	3	1
B	7	3	1
C	8	1	1
D	4	2	1
E	2	4	1
F	10	1	1
G	12	1	1
H	4	6	1

switches to the constraint set (III) and solves the relaxed super-efficiency SBM model. And thus a different reference point, which is Pareto-efficient, is identified by the constraint set (III). However, the assumption that the equalities will not hold if the reference point is weakly efficient might not always be true. In the subsequent, will provide a counter example where the weakly efficient reference point meets the equalities constraints in (II). So the J-SBM model will not switch to the constraint set (III) and the reference point remains weakly efficient.

Consider the data set in Table 2 whose graphic illustration is presented in Fig. 1. The reference point for DMU E identified by S-SBM model is (4, 4, 1). The super-efficiency score is 1.5. The lambda weights of the reference set are $\lambda_D = \lambda_H = 0.5$, $\lambda_A = \lambda_B = \lambda_C = \lambda_F = \lambda_G = 0$, which means the reference point can be generated by combining DMU D and DMU H. Obviously, (4, 4, 1) is not Pareto-efficient because it is dominated by DMU D (4, 2, 1). Since the reference point (4, 4, 1) meets the constraint set (II), we will have the same result if the S-SBM model is employed to evaluate the DMU E. The reference point for DMU E identified by J-SBM is still (4, 4, 1), which is not Pareto-efficient. Also, the super-efficiency score is 1.5 and the lambda weights are $\lambda_D = \lambda_H = 0.5$, $\lambda_A = \lambda_B = \lambda_C = \lambda_F = \lambda_G = 0$. This is contradictory to the corollary 1 in Chen (2013). The corollary says that the reference point identified by the J-SBM model is Pareto-efficient.

III. CONCLUDING REMARKS

One of the main contributions of the work in Chen (2013) is that the reference point identified by the J-SBM model is Pareto-efficient, which is stated in corollary 1 in Chen (2013). In this note, we have shown that the corollary is incorrect by providing a counter example. The reason why the J-SBM cannot guarantee that the reference point identified is Pareto-efficient is that a weakly efficient reference point might still meet the constraint set (II). If the constraint set (II) is met, the J-SBM model works as S-SBM model. Therefore, the J-SBM model might still identify a weakly efficient reference point.

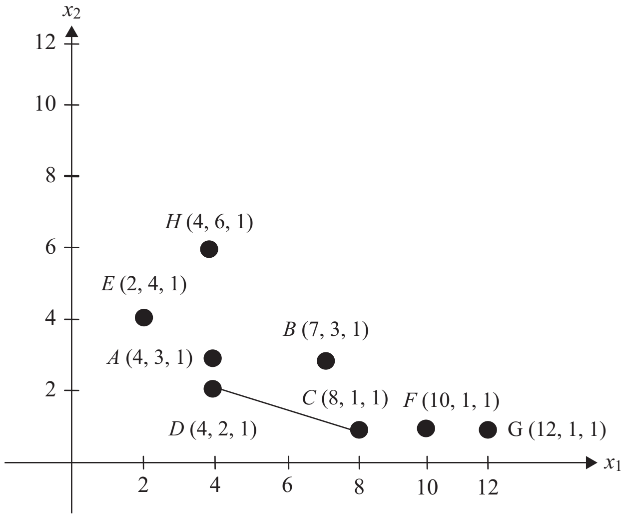

Fig.1. Graphical illustration for DMUs in Table 2.

In Chen (2013), before presenting J-SBM model, Chen provided the following model (model (8) in Chen (2013)) to embed SBM and S-SBM into one model:

$$\min \rho_k^{'} = \frac{1-\frac{1}{m}[\sum_{i=1}^{m} b(s_i^-/x_{ki}) - \sum_{i=1}^{m}(1-b)(s_i^-/x_{ki})]}{1+\frac{1}{s}[\sum_{r=1}^{s} b(s_r^+/y_{kr}) - \sum_{r=1}^{s}(1-b)(s_r^+/y_{kr})]}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1, j\neq k}^{n} \lambda_j x_{ji} \le x_{ki} - bs_i^- + (1-b)s_i^-,\ i = 1, \ldots, m \\ & \sum_{j=1, j\neq k}^{n} \lambda_j y_{jr} \ge y_{kr} + bs_r^+ - (1-b)s_r^+,\ r = 1, \ldots, s \\ & \lambda_j \ge 0,\ j = 1, \ldots, n \\ & s_i^- \ge 0,\ i = 1, \ldots, m \\ & s_r^+ \ge 0,\ r = 1, \ldots, s \\ & b \in \{0, 1\} \end{aligned} \tag{5}$$

As noted by Chen (2013), model (5) switches between SBM and S-SBM. Model (5) can be used to calculate SBM and S-SBM efficiency scores. However, the reference point identified by (5) may still weakly efficient. Therefore, Chen (2013) developed the more advanced model J-SBM so that the reference point identified is Pareto-efficient. However, we have shown that the J-

SBM model still has the same problem. Therefore, model (5) will suffice for embedding SBM and S-SBM.

However, if $s_r^+ > y_{kr}$ when $b = 0$, the objective function in (5) will be negative and unbounded. Since $s_r^+ > y_{kr}$ and s_r^+ approaches to y_{kr} when $b = 0$, the objective function in (5) will go to negative infinity. One more thing has to be noted on (5) is that model (5) needs a small adjustment by adding one constraint, which prevents the objective function from becoming negative when $b = 0$:

$$\min \quad \rho_k' = \frac{1-\frac{1}{m}[\sum_{i=1}^{m} b(s_i^-/x_{ki}) - \sum_{i=1}^{m}(1-b)(s_i^-/x_{ki})]}{1+\frac{1}{s}[\sum_{r=1}^{s} b(s_r^+/y_{kr}) - \sum_{r=1}^{s}(1-b)(s_r^+/y_{kr})]}$$

$$\begin{aligned} s.t. \quad & \sum_{j=1, j\neq k}^{n} \lambda_j x_{ji} \le x_{ki} - bs_i^- + (1-b)s_i^-, \; i = 1, \ldots, m \\ & \sum_{j=1, j\neq k}^{n} \lambda_j y_{jr} \ge y_{kr} + bs_r^+ - (1-b)s_r^+, \; r = 1, \ldots, s \\ & \lambda_j \ge 0, \; j = 1, \ldots, n \\ & s_i^- \ge 0, \; i = 1, \ldots, m \\ & s_r^+ \ge 0, \; r = 1, \ldots, s \\ & s_r^+ \le y_{kr}, \; r = 1, \ldots, s \\ & b \in \{0, 1\} \end{aligned} \tag{6}$$

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