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FAST-CONVERGENCE ITERATIVE ALGORITHMS FOR SOLVING A NONLINEAR BEAM EQUATION WITH AN INTEGRAL TERM SUBJECTED TO DIFFERENT BOUNDARY CONDITIONS

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Key words: nonlinear beam equation, integral term, weak-form method, iterative algorithm.

ABSTRACT

In this paper, a nonlinear beam equation containing an integral term of the deformation energy, which is unknown before the solution is found, is investigated under different boundary conditions. First, we set the unknown integral term as a scalar variable and then develop a weak-form integral equation to solve the integral. By using the sinusoidal functions as test functions and bases of the numerical solution, we obtain a fast-convergence iterative scheme. Due to the orthogonality of the sinusoidal functions, the expansion coefficients of the numerical solution are in the closed form. The proposed iterative algorithms converge quickly and provide highly accurate numerical solutions of the nonlinear beam equation containing the integral term, as confirmed using five numerical examples.

I. INTRODUCTION

In engineering design, beams have a role in nonlinear environments and when subjected to nonlinear conditions (Ma and da Silva, 2004; Dang and Luan, 2010; Dang et al., 2010; Dang and Huong, 2013). Moreover, this problem arises in the study of the transverse vibration of a hinged beam (Woinowski-Krieger, 1960; Feireisl, 1992; Dang and Luan, 2010):

$$y^{(iv)}(x) - \left[\alpha + b_0 \int_0^\ell y'(x)^2\,dx\right] y''(x) = p(x),\ 0 < x < \ell, \quad (1)$$

$$y(0) = y(\ell) = 0, \quad (2)$$

$$y''(0) = y''(\ell) = 0, \quad (3)$$

where α and b_0 are constants and $p(x)$ is an external load. The boundary conditions (BCs) are the simply supported type. In this paper, we develop a fast-convergence iterative method for solving the fourth-order nonlinear ordinary differential equation (ODE) for a nonlinear beam, with an integral term employed.

In previous studies, weak-form integral equations have been successfully used with different test functions and trial functions for solving ODEs. Liu et al. (2017) developed the weak-form integral equation method (WFIEM) to determine the singular solution to problems. The WFIEM and exponentially and polynomially fitted trial solutions, which are designed to automatically satisfy the BCs, can provide an accurate numerical solution of the singular beam equation. Liu et al. (2016) transformed the linear ODE of motion into a linear parabolic-type partial differential equation and then used Green's second identity to derive a boundary integral equation in terms of the adjoint Trefftz test functions. The advantages of this transformation to a weak-form integral formulation were observed when Liu et al. tested some nonlinear inverse vibration problems with a long time span and under high noise. Moreover, Liu (2016) utilized the WFIEM to determine the singular solution. The WFIEM provided accurate and stable solutions for highly singular third-order problems. Liu and Li (2017) employed a fast iterative method to solve a nonlinear beam equation under nonlinear BCs of moments and obtained highly accurate results. In this paper, we extend this idea by using the sinusoidal functions as test functions and bases for developing a powerful beam solver for a nonlinear beam equation containing an integral term of the deformation energy.

II. TEST FUNCTIONS

To solve Eqs. (1)-(3), our first step is to define the integral term as an unknown variable:

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$$\beta := b_0 \int_0^\ell y'(x)^2\, dx. \tag{4}$$

Then, we derive an iterative sequence to determine β_k, which converges to β.

Thus, we have a new linear-like fourth-order ODE, which is given as follows:

$$y^{(iv)}(x) - (\alpha + \beta) y''(x) = p(x),\ 0 < x < \ell. \tag{5}$$

Let

$$v(x, j) := \sin \frac{j\pi x}{\ell},\ j \in \mathbb{N} \tag{6}$$

be the test functions.

By multiplying Eq. (5) by $v(x, j)$ and integrating the resultant from $x = 0$ to ℓ, we obtain the following equation:

$$\int_0^\ell y^{(iv)}(x) v(x, j) dx - (\alpha + \beta) \int_0^\ell y''(x) v(x, j) dx = \int_0^\ell v(x, j) p(x) dx. \tag{7}$$

Integrating by parts the first integral term on the left-hand side four times and using Eqs. (2) and (3), we obtain the following:

$$\begin{aligned}
&\int_0^\ell y^{(iv)}(x) v(x, j) dx \\
&= y'''(x) v(x, j) \Big|_0^\ell - \int_0^\ell y'''(x) v'(x, j) dx \\
&= -\int_0^\ell y'''(x) v'(x, j) dx \text{ due to } v(0, j) = v(\ell, j) = 0 \\
&= \int_0^\ell y''(x) v''(x, j) dx \text{ due to } y''(0) = y''(\ell) = 0 \\
&= -\int_0^\ell y'(x) v'''(x, j) dx \text{ due to } v''(0, j) = v''(\ell, j) = 0 \\
&= \int_0^\ell y(x) v^{(iv)}(x, j) dx \text{ due to } y(0) = y(\ell) = 0.
\end{aligned} \tag{8}$$

Furthermore, we obtain:

$$\begin{aligned}
\int_0^\ell y''(x) v(x, j) dx &= -\int_0^\ell y'(x) v'(x, j) dx \text{ due to } v(0, j) \\
&= v(\ell, j) = 0 \\
&= \int_0^\ell y(x) v''(x, j) dx \text{ due to } y(0) \\
&= y(\ell) = 0.
\end{aligned} \tag{9}$$

Then, we can derive a simple weak-form integral equation to solve $y(x)$ in Eqs. (1)-(3).

Theorem 1. For the nonlinear problem, Eqs. (1)-(3), the solution $y(x)$, β, and the given function $p(x)$ satisfy the following integral equations:

$$\left[\left(\frac{j\pi}{\ell} \right)^4 + (\alpha + \beta) \left(\frac{j\pi}{\ell} \right)^2 \right] \int_0^\ell y(x) v(x, j) dx = \int_0^\ell v(x, j) p(x) dx,\ j \in \mathbb{N}. \tag{10}$$

Proof:

From Eq. (6), we obtain the following:

$$v''(x, j) = -\left(\frac{j\pi}{\ell} \right)^2 v(x, j),\ v^{(iv)}(x, j) = \left(\frac{j\pi}{\ell} \right)^4 v(x, j).$$

Substituting Eqs. (8) and (9) into Eq. (7) and using the aforementioned equation, we derive Eq. (10).

III. TWO SIMPLE ITERATIVE ALGORITHMS

For a simply supported beam, the trial functions in Eq. (6) automatically satisfy the BCs given in Eqs. (2) and (3). Hence, the trial solution of the simply supported beam is as follows:

$$y(x) = \sum_{k=1}^m c_k \sin \frac{k\pi x}{\ell}. \tag{11}$$

To determine the expansion coefficients [c_j ($j = 1, \ldots, m$)], we substitute Eq. (11) into Eq. (10); let $j = 1, \ldots, m$; and use the orthogonality of the following equation:

$$\int_0^\ell \sin \frac{j\pi x}{\ell} \sin \frac{k\pi x}{\ell} dx = \frac{\ell}{2} \delta_{jk}, \tag{12}$$

where δ_{jk} is the Kronecker delta symbol. Consequently, we can derive the following equation:

$$c_j = \frac{2}{\ell \left[\left(\frac{j\pi}{\ell} \right)^4 + (\alpha + \beta) \left(\frac{j\pi}{\ell} \right)^2 \right]} \int_0^\ell v(x, j) p(x) dx,\quad j = 1, \ldots, m. \tag{13}$$

We can also solve Eqs. (1)-(3) by deriving an iterative algorithm to determine β, such that the sequence β_k converges to the true value of β.

Two types of iterative algorithms can be developed. The first iterative algorithm is extremely simple and is described as follows:

(i) Define m, ε_1, and an initial estimate of β, for example $\beta_0 = 0$.

(ii) For $k = 0, 1, \ldots$, calculate the following:

$$c_j = \frac{2}{\ell\left[\left(\frac{j\pi}{\ell}\right)^4 + (\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]}\int_0^\ell v(x, j)p(x)dx,$$

$$j = 1, \ldots, m,$$

and

$$y'(x) = \sum_{j=1}^{m} c_j \frac{j\pi}{\ell}\cos\frac{j\pi x}{\ell},$$

(iii) Substitute the expression for $y'(x)$ into Eq. (4) and calculate the following:

$$\beta_{k+1} = b_0 \int_0^\ell y'(x)^2 dx.$$

If the following convergence criterion,

$$|\beta_{k+1} - \beta_k| \le \varepsilon_1, \tag{14}$$

is satisfied, the iteration is stopped. If the criterion is not satisfied, step (ii) of the next iteration is performed.

The iterative process of the second iterative algorithm is described as follows:

(i) Define m, ε_2, and an initial estimate of c, for example $c_0 = 0$.
(ii) For $k = 0, 1, \ldots$, calculate the following:

$$y'(x) = \sum_{j=1}^{m} c_j \frac{j\pi}{\ell}\cos\frac{j\pi x}{\ell},$$

and

$$\beta = b_0 \int_0^\ell y'(x)^2 dx.$$

(iii) Substitute the expression for β into Eq. (13) and calculate c_j as follows:

$$c_j = \frac{2}{\ell\left[\left(\frac{j\pi}{\ell}\right)^4 + (\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]}\int_0^\ell v(x, j)p(x)dx,$$

$$j = 1, \ldots, m.$$

If the following convergence criterion,

$$\|\mathbf{c}_{k+1} - \mathbf{c}_k\| = \sqrt{\sum_{j=1}^{m}\left(c_j^{k+1} - c_j^k\right)^2} \le \varepsilon_2, \tag{15}$$

is satisfied, the iterations are stopped. If the criterion is not satisfied, perform step (ii) of the next iteration.

The two described algorithms are different. One algorithm generates the sequence of β_k directly, whereas the other generates the sequence of c_k. When c_k is known, we can substitute it into Eq. (4) to determine the numerical solution of $y(x)$.

IV. NUMERICAL EXAMPLES OF SIMPLY SUPPORTED BEAM

1. Example 1

In this example, we consider the simply supported beam example of Dang (2010):

$$\beta = \frac{2}{\pi}\int_0^\pi y'(x)^2 dx, \; p(x) = -4\sin x. \tag{16}$$

We employ $\alpha = 2$, the same as that used by Dang (2010). The exact solution in $x \in [0, \pi]$ is as follows:

$$y(x) = -\sin x. \tag{17}$$

The exact value of β is 1.

For $m = 10$ and $\varepsilon_1 = 10^{-10}$, the first iterative algorithm converges after 36 iterations, as indicated by the solid line in Fig. 1(a). By comparing the numerical and exact solutions, the maximum numerical error in $y(x)$ is 9.37×10^{-12}, as indicated by the solid line in Fig. 1(b), and the absolute error in β is 3.76×10^{-11}.

For $m = 50$ and $\varepsilon_2 = 10^{-10}$, the second iterative algorithm converges after 35 iterations, as indicated by the dashed line in Fig. 1(a). By comparing the numerical and exact solutions, the maximum numerical error in $y(x)$ is 1.88×10^{-11}, as indicated by the dashed line in Fig. 1(b), and the absolute error in β is 7.51×10^{-11}.

2. Example 2

In this example, we consider the following problem (Dang, 2010):

$$\beta = \frac{2}{\pi}\int_0^\pi y'(x)^2 dx, \; p(x) = -x^2. \tag{18}$$

Again, we employ $\alpha = 2$, the same as that used by Dang (2010). No closed-form solution to the problem exists.

For $m = 50$ and $\varepsilon_1 = 10^{-10}$, the first iterative algorithm converges after 33 iterations, as indicated by the solid line in Fig. 2(a). The numerical solution is represented by the solid line in Fig. 2(b). The value of β is 0.94976254, which is similar to the values obtained by Dang (2010) and Shin (1998).

V. NUMERICAL ALGORITHM FOR THE TWO-END FIXED BEAM

Here, we consider a two-end fixed beam subjected to the

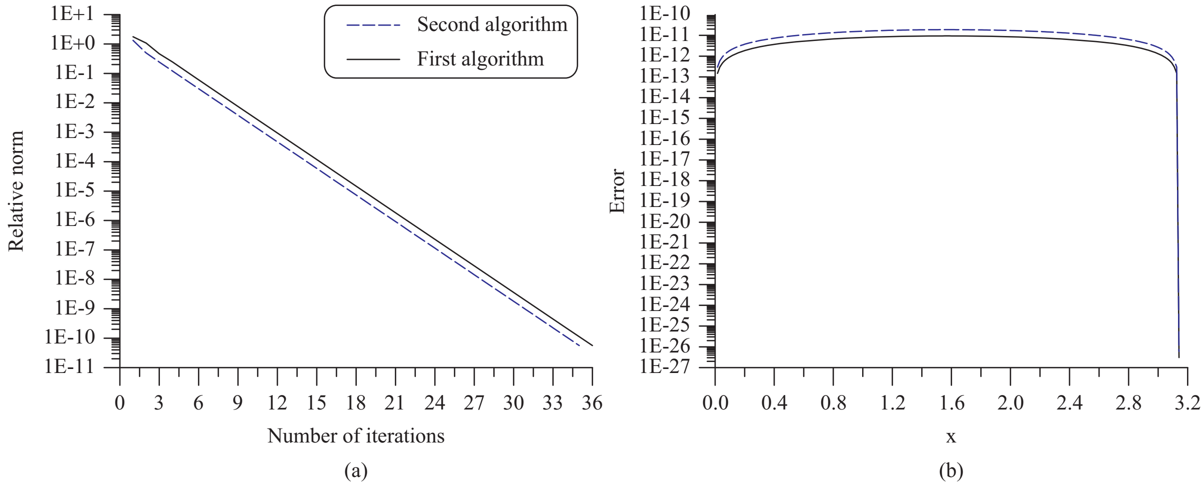

Fig. 1. (a) Convergence rates and (b) numerical errors for the simply supported nonlinear hinged beam.

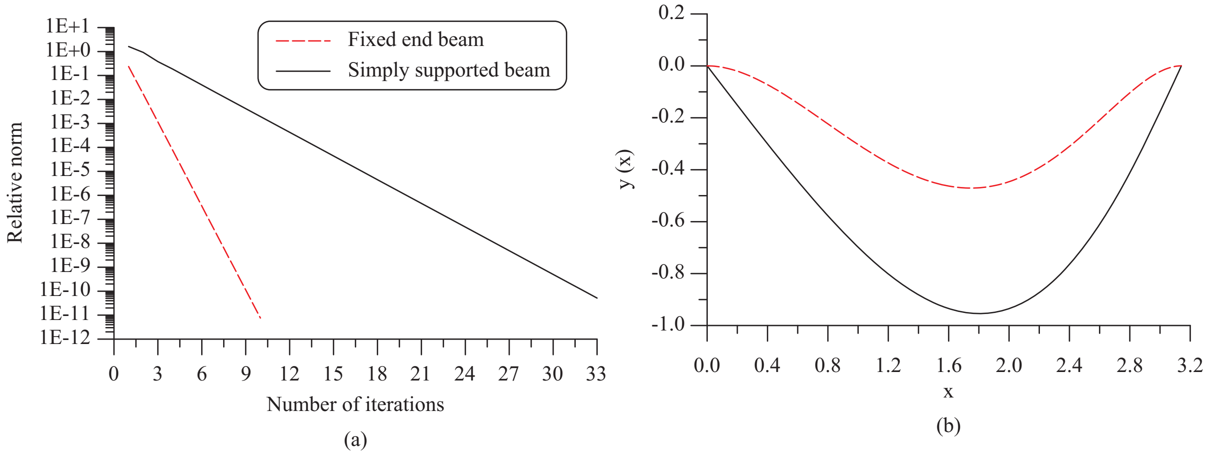

Fig. 2. (a) Convergence rates and (b) numerical solutions for the simply supported and fixed-end beams of examples 2 and 4, respectively.

following BCs:

$$y(0) = y(\ell) = 0, \tag{19}$$

$$y'(0) = y'(\ell) = 0. \tag{20}$$

1. Basic Equations of the Two-End Fixed Beam

Integrating by parts the first integral term on the left-hand side of Eq. (7) four times and using Eqs. (19) and (20), we obtain the following equation:

$$\begin{aligned}
&\int_0^\ell y^{(iv)}(x)v(x, j)dx = y'''(x)v(x, j)\Big|_0^\ell - \int_0^\ell y'''(x)v'(x, j)dx \\
&= -\int_0^\ell y'''(x)v'(x, j)dx \text{ due to } v(0, j) = v(\ell, j) = 0 \\
&= -y''(\ell)v'(\ell, j) + y''(0)v'(0, j) + \int_0^\ell y''(x)v''(x, j)dx \\
&= -y''(\ell)v'(\ell, j) + y''(0)v'(0, j) \\
&\qquad - \int_0^\ell y'(x)v'''(x, j)dx \text{ due to } v''(0, j) \\
&= v''(\ell, j) = 0 \\
&= -y''(\ell)v'(\ell, j) + y''(0)v'(0, j) \\
&\qquad - \int_0^\ell y'(x)v'''(x, j)dx \text{ due to } v''(0, j) \\
&= v''(\ell, j) = 0.
\end{aligned} \tag{21}$$

Therefore, we can derive a simple weak-form integral equation for $y(x)$ and $p(x)$ to solve Eqs. (1), (19), and (20) for the fixed-end beam.

Theorem 2. In the problem described by Eqs. (1), (19), and (20), $y(x)$, β, and $p(x)$ satisfy the following integral equations:

$$\left[\left(\frac{j\pi}{\ell}\right)^4+(\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]\int_0^\ell y(x)v(x,\,j)dx$$
$$= y''(\ell)v'(\ell,\,j)-y''(0)v'(0,\,j)+\int_0^\ell v(x,\,j)p(x)dx,\ j\in\mathbb{N}. \tag{22}$$

Proof.

By substituting $v^{(iv)}(x, j) = (j\pi)^4 v(x, j)/\ell^4$ into Eq. (21) and equating Eq. (9) to Eq. (7), we prove Theorem 2.

In Eq. (22), we can expand the solution of $y(x)$ as follows:

$$y(x)=c_{m+1}x\left(1-\frac{x}{\ell}\right)+c_{m+2}x^2\left(1-\frac{x}{\ell}\right)+\sum_{k=1}^m c_k\sin\frac{k\pi x}{\ell}, \tag{23}$$

which automatically satisfies the first BC [Eq. (19)]. Here, the expansion coefficients are unknown [c_j (j = 1, …, m + 2)] and must be determined. From Eq. (23), we obtain the following equation:

$$y'(x)=c_{m+1}\left(1-\frac{2x}{\ell}\right)+c_{m+2}\left(2x-\frac{3x^2}{\ell}\right)+\sum_{k=1}^m c_k\frac{k\pi}{\ell}\cos\frac{k\pi x}{\ell}. \tag{24}$$

From the second BC [Eq. (20)], we obtain the following equations:

$$c_{m+1}+\sum_{k=1}^m c_k\frac{k\pi}{\ell}=0, \tag{25}$$

$$-c_{m+1}-\ell c_{m+2}+\sum_{k=1}^m c_k\frac{k\pi}{\ell}\cos(k\pi)=0. \tag{26}$$

To determine the expansion coefficients, we must derive m other algebraic equations. By substituting Eq. (23) into Eq. (22), letting j = 1, …, m, and using the orthogonality in Eq. (12), we can derive the following equation:

$$\frac{\ell}{2}c_j+c_{m+1}\int_0^\ell x\left(1-\frac{x}{\ell}\right)v(x,\,j)dx+c_{m+2}\int_0^\ell x^2\left(1-\frac{x}{\ell}\right)v(x,\,j)dx$$
$$=\frac{1}{\left(\frac{j\pi}{\ell}\right)^4+(\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2}[y''(\ell)v'(\ell,\,j)-y''(0)v'(0,\,j)$$
$$+\int_0^\ell v(x,\,j)p(x)dx]. \tag{27}$$

Eqs. (25)-(27) can be arranged into an (m + 2)-dimensional linear system as follows:

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}c_1\\ \vdots\\ c_m\\ c_{m+1}\\ c_{m+2}\end{bmatrix}=\begin{bmatrix}e_1\\ \vdots\\ e_m\\ 0\\ 0\end{bmatrix}. \tag{28}$$

This system can be used to determine the expansion coefficients [c_k (k = 1,…, m + 2)], where

$$e_j:=\frac{1}{\left(\frac{j\pi}{\ell}\right)^4+(\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2}\int_0^\ell v(x,\,j)p(x)dx. \tag{29}$$

The dimensions of **A**, **B**, **C**, and **D** are $m \times m$, $m \times 2$, $2 \times m$, and 2×2, respectively. The components of these matrices are as follows:

$$A_{jk}=\frac{\ell}{2}\delta_{jk},\ j,k=1,\ldots m,$$
$$B_{j1}=\int_0^\ell x\left(1-\frac{x}{\ell}\right)v(x,\,j)dx$$
$$+\frac{2}{\ell\left[\left(\frac{j\pi}{\ell}\right)^4+(\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]}\left[v'(\ell,\,j)-v'(0,\,j)\right],$$
$$j=1,\ldots,m,$$
$$B_{j2}=\int_0^\ell x^2\left(1-\frac{x}{\ell}\right)v(x,\,j)dx$$
$$+\frac{2}{\left(\frac{j\pi}{\ell}\right)^4+(\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2}\left[2v'(\ell,\,j)-v'(0,\,j)\right],$$
$$j=1,\ldots,m,$$
$$C_{1j}=\frac{j\pi}{\ell},\ j=1,\ldots,m,$$
$$C_{2j}=\frac{j\pi\cos j\pi}{\ell},\ j=1,\ldots,m,$$
$$D_{11}=1,\ D_{12}=0,\ D_{21}=-1,\ D_{22}=-\ell. \tag{30}$$

We can exactly solve Eq. (28) by using the Drazin inversion formula.

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1}=\begin{bmatrix}\frac{2}{\ell}\mathbf{I}_M+\frac{4}{\ell^2}\mathbf{B}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} & -\frac{2}{\ell}\mathbf{B}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\\ -\frac{2}{\ell}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} & \left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\end{bmatrix}. \tag{31}$$

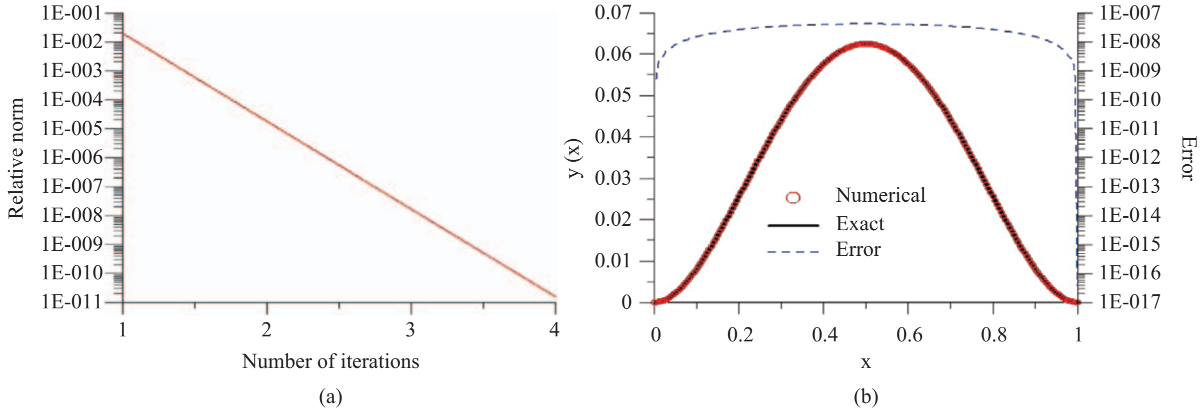

Fig. 3. (a) Convergence rates and (b) displacement and numerical errors for the fixed-end nonlinear hinged beam.

From the Drazin inversion formula, we obtain the following:

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \\ c_{m+1} \\ c_{m+2} \end{bmatrix} = \begin{bmatrix} \frac{2}{\ell}\mathbf{I}_M + \frac{4}{\ell^2}\mathbf{B}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} & -\frac{2}{\ell}\mathbf{B}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1} \\ -\frac{2}{\ell}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} & \left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_m \\ e_{m+1} \\ e_{m+2} \end{bmatrix}. \quad (32)$$

Because $e_{m+1} = e_{m+2} = 0$, Eq. (32) can be simplified to

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \\ c_{m+1} \\ c_{m+2} \end{bmatrix} = \begin{bmatrix} \frac{2}{\ell}\mathbf{I}_M + \frac{4}{\ell^2}\mathbf{B}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} \\ -\frac{2}{\ell}\left(\mathbf{D}-\frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}. \quad (33)$$

The two types of iterative algorithms thus described are similar to the algorithms described in Section 3.

2. Example 3

In this example, we consider the following equation:

$$p(x) = -12x^2 + 12x + 22. \quad (34)$$

The exact solution in $x \in [0, 1]$ is

$$y(x) = x^4 - 2x^3 + x^2. \quad (35)$$

In this example, the values of α and b_0 in Eq. (1) are taken as 103/105 and 1, respectively. The exact value of β obtained is 2/105.

For $m = 100$ and $\varepsilon_1 = 10^{-10}$, the iterative scheme converges after four iterations, as displayed in Fig. 3(a). The numerical error is represented by the dashed line in Fig. 3(b), and its maximum value is 4.02×10^{-8}. The absolute error inf β is 2.07×10^{-8}.

3. Example 4

In this example, we consider the following equation:

$$\beta = \frac{2}{\pi}\int_0^{\pi} y'(x)^2\,dx,\ p(x) = -x^2. \quad (36)$$

We employ $\alpha = 2$, the same as that used by Dang (2010). No close-form solution exists. For $m = 50$ and $\varepsilon_1 = 10^{-10}$, the first iterative algorithm converges after 10 iterations, as indicated by the dashed line in Fig. 2(a). The numerical solution is represented by a dashed line in Fig. 2(b), and $\beta = 0.222553$ is determined. Under the same load, the deflection of the fixed-end beam is much smaller than that of the simply supported beam.

VI. CANTILEVER BEAM AND EXAMPLES

1. Basic Equations for a Cantilever Beam

For the cantilever beam, we have the following BCs:

$$y(0) = y'(0) = 0, \quad (37)$$

$$y''(\ell) = y'''(\ell) = 0. \quad (38)$$

The trial solution for a cantilever beam is as follows:

$$y(x) = b_{m+1}x + b_{m+2}\left(\frac{\ell x^2}{2} - \frac{x^3}{6}\right) + \sum_{k=1}^{m} b_k \sin\frac{k\pi x}{\ell}. \quad (39)$$

In this equation, the conditions $y(0) = 0$ and $y''(\ell) = 0$ are automatically satisfied.

Through a simple derivation, we can obtain

$$\int_0^{\ell} y^{(iv)}(x)v(x, j)dx = y'''(x)v(x, j)\Big|_0^{\ell} - \int_0^{\ell} y'''(x)v'(x, j)dx$$

$$= -\int_0^{\ell} y'''(x)v'(x, j)dx \text{ due to } v(0, j) = v(\ell, j) = 0$$

$$= y''(0)v'(0, j) + \int_0^\ell y''(x)v''(x, j)dx \text{ due to } y''(\ell) = 0$$

$$= y''(0)v'(0, j) - \int_0^\ell y'(x)v'''(x, j)dx \text{ due to } v''(0, j) = v''(\ell, j) = 0$$

$$= y''(0)v'(0, j) - y(\ell)v'''(\ell, j)$$

$$+ \int_0^\ell y(x)v^{iv}(x, j)dx \text{ due to } y(0) = 0.$$

Then, we can derive the following result:

Theorem 3. In the problem described by Eqs. (1), (37), and (38), $y(x)$, β, and $p(x)$ satisfy the following integral equations:

$$\left[\left(\frac{j\pi}{\ell}\right)^4 + (\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]\int_0^\ell y(x)v(x, j)dx + (\alpha+\beta)y(\ell)v'(\ell, j) = y(\ell)v'''(\ell, j) - y''(0)v'(0, j) + \int_0^\ell v(x, j)p(x)dx,\ j \in \mathbb{N}. \tag{40}$$

From Eq. (39), we obtain the following:

$$y'(x) = b_{m+1} + b_{m+2}\left(\ell x - \frac{x^2}{2}\right) + \sum_{k=1}^m b_k \frac{k\pi}{\ell}\cos\frac{k\pi x}{\ell}, \tag{41}$$

$$y''(x) = b_{m+2}(\ell - x) - \sum_{k=1}^m b_k \left(\frac{k\pi}{\ell}\right)^2 \sin\frac{k\pi x}{\ell}, \tag{42}$$

$$y'''(x) = -b_{m+2} - \sum_{k=1}^m b_k \left(\frac{k\pi}{\ell}\right)^3 \cos\frac{k\pi x}{\ell}. \tag{43}$$

Because the conditions $y(0) = 0$ and $y''(\ell) = 0$ are automatically satisfied, we only impose two other BCs [$y'(0) = 0$ and $y'''(\ell) = 0$]:

$$b_{m+1} - \sum_{k=1}^m b_k \frac{k\pi}{\ell} = 0, \tag{44}$$

$$b_{m+2} - \sum_{k=1}^m b_k \left(\frac{k\pi}{\ell}\right)^3 \cos(k\pi) = 0. \tag{45}$$

By the same logic, we obtain the following:

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \\ b_{m+1} \\ b_{m+2} \end{bmatrix} = \begin{bmatrix} \frac{2}{\ell}\mathbf{I}_M + \frac{4}{\ell^2}\mathbf{B}\left(\mathbf{D} - \frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} \\ -\frac{2}{\ell}\left(\mathbf{D} - \frac{2}{\ell}\mathbf{CB}\right)^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{46}$$

where

$$e_j : \frac{1}{\left(\frac{j\pi}{\ell}\right)^4 + (\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2}\int_0^\ell v(x, j)p(x)dx,\ j = 1, \ldots m,$$

$$B_{j1} = \int_0^\ell xv(x, j)dx - \frac{\ell}{\left[\left(\frac{j\pi}{\ell}\right)^4 + (\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]}\int_0^\ell v'''(\ell, j) - (\alpha+\beta)v'(\ell, j),$$

$$j = 1, \ldots m$$

$$B_{j2} = \int_0^\ell \left(\frac{\ell x^2}{2} - \frac{x^3}{6}\right)v(x, j)dx + \frac{\ell}{\left[\left(\frac{j\pi}{\ell}\right)^4 + (\alpha+\beta)\left(\frac{j\pi}{\ell}\right)^2\right]}\left[v'(0, j)\frac{\ell^2}{3}\left[v'''(\ell, j) - (\alpha+\beta)v'(\ell, j)\right]\right],$$

$$j = 1, \ldots, m,$$

$$C_{1j} = \frac{j\pi}{\ell},\ C_{2j} = \frac{(j\pi)^3\cos j\pi}{\ell^3},\ j = 1, \ldots, m,\ \mathbf{D} = \mathbf{I}_2. \tag{47}$$

2. Example 5

In example 5, we consider the following:

$$p(x) = 48x - 24x^2. \tag{48}$$

The exact solution in $x \in [0, 1]$ is as follows:

$$y(x) = x^4 - 4x^3 + 6x^2. \tag{49}$$

Here, the values of α and b_0 in Eq. (1) are taken as 1 and 7/72, respectively. The exact value of β is determined to be 1.

For $m = 150$ and $\varepsilon_1 = 10^{-9}$, the iterative scheme converges after 11 iterations, as displayed in Fig. 4(a). The numerical error is represented by a dashed line in Fig. 4(b), and its maximum error is 1.49×10^{-2}. The absolute error in β is 9.98×10^{-3}.

VII. CONCLUSIONS

In this paper, we derived simple and effective iterative methods for numerically solving nonlinear beam problems that include an unknown deformation energy term in the governing equation. We introduced different linear systems in the solutions by subjecting the simply supported beam, two-end fixed beam, and cantilever beam to different BCs. The closed-form coefficients in the numerical solution were obtained by using the orthogonal property of test functions and the Drazin inversion formula. These coefficients reduced the computational complexity of the problems and increased the speed of convergence of the proposed algorithm. Five numerical examples were used to examine the stability and accuracy of the presented iterative algorithms.

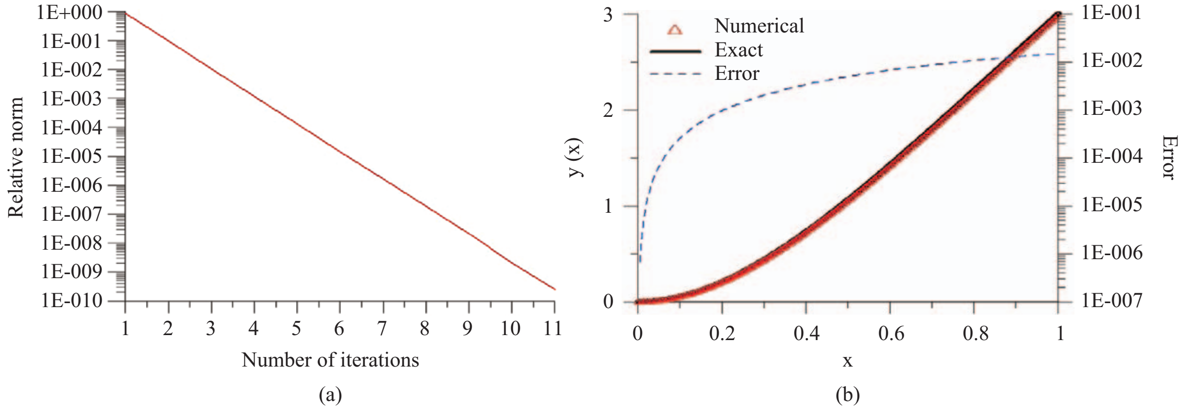

Fig. 4. (a) Convergence rates and (b) displacement and numerical error for the cantilever nonlinear beam.

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