Journal of Marine Science and Technology, Vol. 27, No. 3, pp. 202-209 (2019)
DOI: 10.6119/JMST.201906_27(3).0002

IDENTIFYING NONLINEAR OSCILLATORS BY AN ENERGETIC FUNCTIONAL IN THE LINEAR SPACE OF TEMPORAL BOUNDARY FUNCTIONS

Chein-Shan Liu[1, 4], Yung-Wei Chen[2, 4], and Jiang-Ren Chang[3, 4]

Key words: inverse nonlinear oscillator problem, energetic functional, energetic temporal boundary functions, iterative method.

ABSTRACT

We resolve the inverse problems of a second-order nonlinear oscillator to recover time-dependent damping function and nonlinear restoring force, with the help of temporal boundary data measured at initial time and final time. By using these data, a sequence of temporal boundary functions of time is derived, which satisfy the measured temporal boundary conditions automatically, and are at least the fourth-order polynomials of time. All the temporal boundary functions and zero element constitute a linear space, and a new concept of energetic functional is introduced in the linear space, of which the energy is preserved for each energetic temporal boundary function. We employ the energetic temporal boundary functions as the bases of numerical solutions. Then, the linear systems are derived and the iterative algorithms used to recover the unknown nonlinear oscillators are developed from the energetic functional, which are convergent very fast. We can recover the damping functions and restoring forces of nonlinear oscillators, among them the nonlinear ship rolling oscillator and the Duffing nonlinear oscillator are of tested examples. The required data are parsimonious, merely the measured temporal boundary data of displacement and velocity, and the temporal boundary data of unknown function to be recovered.

I. INTRODUCTION

In this paper we are going to develop a very simple mathematical method to treat the following recovery problem of a nonlinear oscillator:

$$\ddot{x}(t) + c(t)\dot{x}(t) + H(x) = F(t), \; t \in (0, a), \tag{1}$$

where $c(t)$ denotes the damping coefficient, $F(t)$ and $H(x)$ are the external excitation and restoring force, and $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ are displacement, velocity, and acceleration of the response of a system, respectively. However, this is a set of index-three differential algebraic equations (Liu, 2008a), which is difficult to solve because the amplification of small errors and perturbations in the displacement result in severe numerically ill-posed conditions. Generally, there are different inverse problems of Eq. (1): (a) identifying the damping coefficient $c(t)$, (b) identifying the nonlinear restoring force $H(x)$, and (c) identifying the unknown forcing function $F(t)$.

The topic (c) has been studied by the author and his co-workers (Liu and Chang, 2016; Liu et al., 2016). In this paper we turn our attention to the topics (a) and (b). For the purpose to solve the topic (a), we supplement the following temporal boundary data of x and c:

$$x(0) = x_0, \; x(a) = x_a, \; \dot{x}(0) = \dot{x}_0, \; \dot{x}(a) = \dot{x}_a, \tag{2}$$

$$c(0) = c_0, \; c(a) = c_a, \; \dot{c}(0) = \dot{c}_0, \; \dot{c}(a) = \dot{c}_a, \tag{3}$$

where $c(t)$ depends on time and $\dot{c}_a \neq 0$.

It means that we will use these measured data in Eqs. (2) and (3) to recover the unknown coefficient function $c(t)$ in a time interval of $t \in (0, a)$. Identification of viscous damping and non-viscous damping were presented by Adhikari and Woodhouse (2001a, 2001b). Meanwhile, an iteration method for solving viscoelastic motion with fractional differential operator of damping was also developed (Ingman and Suzdalnisky, 2001). Recently, Liu (2008a) has developed a Lie-group shooting method to identify $c(t)$, which is however resorted on the displacement data measured in a whole time interval. By the same token, we will recover the nonlinear restoring force in Section 4.

The inverse problem is a severely ill-posed problem, since very close input data may correspond to enough different re-

Paper submitted 01/15/19; revised 03/14/19; accepted 03/29/19. Author for correspondence: Jiang-Ren Chang (e-mail: cjr@mail.ntou.edu.tw).
[1] *Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu, China.*
[2] *Department of Marine Engineering, National Taiwan Ocean University, Keelung, Taiwan, R.O.C.*
[3] *Department of Systems Engineering and Naval Architecture, National Tai wan Ocean University, Keelung, Taiwan, R.O.C.*
[4] *Center of Excellence for Ocean Engineering, National Taiwan Ocean University, Keelung, Taiwan, R.O.C.*

sults. The inverse problem related to the determination of the leading coefficient of the Sturm-Liouville equation from the boundary measurements has been studied by Hasanov and Shores (1997), and the numerical methods have been developed by Hasanov and Pektas (2002), Seyidmamedov and Hasanov (2002), Hasanov and Seyidmamedov (2002), and Liu (2011). Basically, the dynamic inverse problem (1)-(3) is more difficult, of which no similar results as that for the Sturm-Liouville equation exists in the literature.

The dissipation of energy in a mechanical system is often described by a viscous damping term and a friction element, while the conservative part is described by a nonlinear spring element. The resulting dynamic equation is useful because it can be used mathematically as a model to simulate the nonlinear oscillation behavior. However, we may encounter the problem that the material properties of structure are not yet known, and then the resulting oscillatory problem is an inverse vibration problem. To identify the damping term and the restoring force of a newly received structure are of great importance for the design, control and stability analysis of machines, vehicles and structural systems, of which the vibration behavior is heavily dependent on the mathematical modelling.

Liu (2008b) has developed a Lie-group shooting method to identify the restoring force, which resorted on the displacement data in a whole time interval. The conjugate gradient method to estimate the time-dependent stiffness coefficients (Huang, 2001) and balancing energy technique (Liang and Feeny, 2006) to estimate damping parameters were also proposed. In addition, Kerschen et al. (2006) have given a comprehensive review of the developments of some useful methods in the nonlinear system identification of structural dynamics. The parameter identification is a major step towards the establishment of a structural model with good predictive capability. The restoring force surface method or force state mapping method is a simple procedure allowing a direct identification of restoring force for nonlinear mechanical systems. The basic procedures were introduced by Masri and Caughey (1979), and then extended by Crawley and Aubert (1986), Crawley and O′Donnel (1986), and Duym et al. (1995). Namdeo and Manohar (2008) have identified nonlinear system parameters from the measured time histories of response under known excitations. Although this numerical method has been applied to mechanical experiments, how to ensure numerical stability and avoid noise disturbance are not reported especially for the polynomial function with high-order. In recent years, studies on the developments of model free nonlinear restoring force identification with their numerical and experimental validation have been presented. Some results have attracted much attention of this field (He et al., 2012; Xu et al., 2012).

Usually, in the realm of the inverse vibration problems, the data required are the displacement, velocity and/or acceleration measured in a whole time interval. In this paper we are going to propose a novel approach by using the temporal boundary measurements to recover the damping function and identify the nonlinear oscillator, like that in the identification of the leading coefficient of the Sturm-Liouville problem with boundary data (Hasanov and Shores, 1997). This identification technique if possible would be much data saving and time saving in the solutions of the nonlinear inverse vibration problems.

The remainder of this paper is arranged as follows. In Section 2 we introduce a new concept of the energy functional in terms of temporal boundary functions, which constitute a linear space of all polynomial functions of time with at least the fourth-order, and satisfy the measured temporal boundary conditions. In Section 3 we derive the iterative algorithm to recover the unknown damping functions and two examples are given. In Section 4 we derive the iterative algorithm to recover the unknown restoring forces, and four numerical examples are given for the identification of different type restoring forces. Finally, the conclusions are drawn in Section 5.

II. ENERGETIC FUNCTIONAL OF TEMPORAL BOUNDARY FUNCTIONS

For a linear conservative system:

$$\ddot{x}(t) + kx(t) = 0 ,$$

by multiplying $\dot{x}$ on both sides we have

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{k}{2}x^2\right) = 0 ,$$

$$\int_0^t \left(\ddot{x} + kx\right)\dot{x}(\xi)d\xi = \frac{1}{2}\int_0^t \frac{d}{d\xi}\left(\dot{x}^2 + kx^2\right)d\xi$$

$$= \frac{1}{2}\left[\dot{x}^2(t) - \dot{x}^2(0)\right] + \frac{k}{2}\left[x^2(t) - x^2(0)\right] = 0 ,$$

such that the energy is conserved, i.e.,

$$\frac{1}{2}\dot{x}^2(t) + \frac{k}{2}x^2(t) = \frac{1}{2}\dot{x}_0^2 + \frac{k}{2}x_0^2 = \text{Constant} ,$$

where the initial displacement $x(0) = x_0$ and initial velocity $\dot{x}(0) = \dot{x}_0$ determine the energy of free vibration. When the damping term is added into the above equation the method of balancing energy was used by Liang and Feeny (2006, 2011) to identify the damping parameters.

The energy formulation can be extended to nonlinear systems, and the resulting equation is an energy equation. This motivates us to use the energy functional as a mathematical tool to identify the nonlinear system. Usually, the given data in Eq. (2) are not zero, which leaves us an obstacle to set up a linear space to be introduced below. Before embarking the analysis we seek a variable transformation by

$$y(t) = x(t) - b(t) , \tag{4}$$

where

$$b(t)=\frac{1}{a^3}\left[2x_0-2x_a+\dot{x}_0a+\dot{x}_aa\right]t^3 \\ -\frac{1}{a^2}\left[3x_0-3x_a+2\dot{x}_0a+\dot{x}_aa\right]t^2+\dot{x}_0t+x_0 \tag{5}$$

is a homogenized function of time, such that we have a new system with homogeneous temporal boundary conditions of $y(t)$:

$$\ddot{y}(t)+\ddot{b}(t)+c(t)\left[\dot{y}(t)+\dot{b}(t)\right]+H(y(t)+b(t))=F(t),\ x\in(0,a), \tag{6}$$

$$y(0)=0,\ y(a)=0,\ \dot{y}(0)=0,\ \dot{y}(a)=0. \tag{7}$$

Multiplying Eq. (6) by $\dot{y}(t)+\dot{b}(t)$, integrating it from $t = 0$ to $t = a$ and in view of Eqs. (7) and (2), we can obtain the following energy identity:

$$\int_0^a\left\{c(t)\left[\dot{y}(t)+\dot{b}(t)\right]^2-F(t)\left[\dot{y}(t)+\dot{b}(t)\right]\right\}dt \\ =\frac{1}{2}\left[\dot{x}_0^2+x_a^2\right]+Q(x_0)-Q(x_a):=d_0, \tag{8}$$

where $Q(x)=\int H(x)dx$ and d_0 is a constant. In above, the damping dissipation, the potential energy, the kinetic energy and the external work are balanced. This equation is useful for the reconstruction of the damping coefficient $c(t)$.

Really, we cannot exactly know $y(t)$ in Eqs. (6) and (8), because $c(t)$ is an unknown function to be determined. However, we can set up some functions to approximate $y(t)$. First, we can derive the temporal boundary function which automatically satisfies the temporal boundary conditions in Eq. (7):

$$B_j(t)=(t^4-2at^3+at^2)t^{j-1},\ j\geq 1. \tag{9}$$

They are at least the fourth-order polynomial temporal boundary functions, satisfying the following homogeneous temporal boundary conditions:

$$B_j(0)=0,\ B_j(a)=0,\ \dot{B}_j(0)=0,\ \dot{B}_j(a)=0. \tag{10}$$

From Eqs. (9) and (10) it is obvious that when $B_j(t)$ is a temporal boundary function, $\beta B_j(t)$, $\beta\in\mathbb{R}$ is also a temporal boundary function, and when $B_j(t)$ and $B_k(t)$ are temporal boundary functions, $B_j(t)+B_k(t)$ is also a temporal boundary function. The temporal boundary functions are closure under a scalar multiplication and addition, such that the set of

$$\left\{B_j(t)\right\},\ j\geq 1, \tag{11}$$

and the zero element constitute a linear space of temporal boundary functions, allowing the combination of $B_j(t)$ to be another linear element:

$$E_j(t)=\gamma_jB_j(t)+B_{j+1}(t),\ j\geq 1,\ j \text{ not summed}, \tag{12}$$

which is also an element of the linear space, satisfying

$$E_j(0)=0,\ E_j(a)=0,\ \dot{E}_j(0)=0,\ \dot{E}_j(a)=0. \tag{13}$$

Now the problem is how to determine γ_j for each linear element of $E_j(t)$.

Because $E_j(t)$ already satisfies the temporal boundary conditions in Eq. (13), we turn our attention to the energy identity (8), from which we can approximate $y(t)$ by $E_j(t)$, and obtain

$$\int_0^a\left\{c(t)\left[\dot{E}_j(t)+\dot{b}(t)\right]^2-F(t)\left[\dot{E}_j(t)+\dot{b}(t)\right]\right\}dt=d_0, \tag{14}$$

which is an energetic functional of $E_j(t)$ defined in the linear space.

Inserting Eq. (12) for $E_j(t)$ and

$$\dot{E}_j(t)=\gamma_j\dot{B}_j(t)+\dot{B}_{j+1}(t) \tag{15}$$

into Eq. (14), we can derive a quadratic nonlinear equation for γ_j:

$$a_2\gamma_j^2+a_1\gamma_j+a_0=0, \tag{16}$$

where

$$a_2=\int_0^a c(t)\,\dot{B}_j^2(t)dt, \\ a_1=\int_0^a\left\{2c(t)\left[\dot{B}_{j+1}(t)+\dot{b}(t)\right]\dot{B}_j(t)-F(t)\dot{B}_j(t)\right\}dt, \\ a_0=\int_0^a\left\{c(t)\left[\dot{B}_{j+1}(t)+\dot{b}(t)\right]^2-F(t)\left[\dot{B}_{j+1}(t)+\dot{b}(t)\right]\right\}dt-d_0, \tag{17}$$

The solution of γ_j in Eq. (16) is

$$\gamma_j=\frac{-a_1-\sqrt{a_1^2-4a_0a_2}}{2a_2}. \tag{18}$$

The linear element $E_j(t)$ in Eq. (12) upon endowing with the above γ_j is an energetic temporal boundary function, which not only satisfies the temporal boundary conditions, but also preserves the energy in Eq. (14).

Up to here it is witnessed to determine γ_j by using the energy functional in Eq. (14). Due to this reason, $E_j(t)$ is called an energetic temporal boundary function, and correspondingly the numerical method based on $E_j(t)$ to be introduced below is an energetic temporal boundary function method (ETBFM).

III. RECOVERING THE DAMPING FUNCTIONS

Because the temporal boundary data of $c(t)$ are given in Eq. (3) we can introduce the following translation function of t:

$$
\begin{aligned}
d(t) = &\frac{1}{a^3}\left[2c_0 - 2c_a + \dot{c}_0 a + \dot{c}_a a\right]t^3 \\
&+\frac{1}{a^2}\left[3c_0 - 3c_a + 2\dot{c}_0 a + \dot{c}_a a\right]t^2 + \dot{c}_0 t + c_0,
\end{aligned} \tag{19}
$$

such that we have

$$
d(0) = c_0,\ d(a) = c_a,\ \dot{d}(0) = \dot{c}_0,\ \dot{d}(a) = \dot{c}_a. \tag{20}
$$

We suppose that the unknown damping function $c(t)$ can be expanded by

$$
c(t) = d(t) + \sum_{k=1}^{m} b_k E_k(t), \tag{21}
$$

which automatically satisfies $c(0) = c_0$, $c(a) = c_a$, $\dot{c}(0) = \dot{c}_0$ and $\dot{c}(a) = \dot{c}_a$, due to Eqs. (20) and (13).

Then, we use the above $c(t)$ with an initial guess of b_k to set up the system of linear elements $E_j(t)$ by the method in Section 2. During the iteration process, $E_j(t)$ are modified by $c(t)$, which is varying step-by-step.

We can derive a system of linear algebraic equations by inserting $c(t)$ of Eq. (21) and different $E_j(t)$ with $j = 1, \dots, m$ into Eq. (14):

$$
\begin{aligned}
&b_k \int_0^a \left[\dot{E}_j(t) + \dot{b}(t)\right]^2 \dot{E}_k(t)\, dt \\
&= d_0 + \int_0^a F(t)\left[\dot{E}_j(t) + \dot{b}(t)\right] dt - \int_0^a d(t)\left[\dot{E}_j(t) + \dot{b}(t)\right]^2 dt
\end{aligned} \tag{22}
$$

Solving this linear system we can determine the expansion coefficients b_k, $k = 1, \dots, m$. Then, we can estimate $c(t)$ by Eq. (21).

1. Iterative Algorithm

The numerical procedures of the energetic temporal boundary function method (ETBFM) are summarized as follows.

(1) Give m, ε, and an initial guess of $\mathbf{b} = (b_1, \dots, b_m)^{\mathrm{T}}$,

(2) For $k = 1, \dots,$ calculate

$$
E_j(t) = \gamma_j B_j(t) + B_{j+1}(t),
$$

$$
c(t) = d(t) + \sum_{j=1}^{m} b_j^k E_j(t),
$$

and calculate a_2, a_1 and a_0 in Eq. (17),

(3) Calculate γ_j by Eq. (18),

$$
E_j(t) = \gamma_j B_j(t) + B_{j+1}(t),
$$

$$
\dot{E}_j(t) = \gamma_j \dot{B}_j(t) + \dot{B}_{j+1}(t),
$$

(4) Insert the above $E_j(t)$ and $\dot{E}_j(t)$ into Eq. (22), and solve the linear system to obtain b_j^{k+1}. If the following convergence criterion for the relative norm of $\mathbf{b}^k$ is satisfied:

$$
\left\|\mathbf{b}^{k+1} - \mathbf{b}^k\right\| \le \varepsilon, \tag{23}
$$

then stop the iterations; otherwise, go to (2) to the next step. Notice that $a_1^2 - 4a_0 a_2$ in Eq. (18) may be negative in the first iteration, and we use $\left|a_1^2 - 4a_0 a_2\right|$ to avoid the interruption of program.

2. Recovering *c*(*t*)

In this section we solve the inverse coefficient problem of nonlinear oscillator by recovering the unknown damping function $c(t)$ by using the temporal boundary data.

Example 1

This example is given by

$$
c(t) = (t-3)^2 + \sin(\pi t),\ x(t) = \sin(t),
$$

$$
H(x) = \alpha x + \beta x^3, \tag{24}
$$

where the exact $F(t)$ can be derived according to Eq. (1). The above $H(x)$ is the restoring force of the Duffing nonlinear oscillator.

Under $m = 2$, $\varepsilon = 10^{-3}$, $a = 1$, $\alpha = 5$ and $\beta = 2$, the iterative algorithm converges with 5 steps as shown in Fig. 1(a). Upon comparing with the exact $c(t)$, good result is obtained with the maximum error being 0.154 as shown in Fig. 1(b). Additionally, the different m and ε are considered to test convergence speed and numerical accuracy, and the numerical result is shown in Table 1. Table 1 shows that the maximum error increases from 0.107 to 1.224 when m increases. That is, the numerical error

Table 1. Different m and ε affect the different convergences of the present scheme.

m	ε	Maximum error	Iterative number
2	10^{-3}	0.153	5
4		0.556	10
6		1.224	93
2	10^{-2}	0.107	2
2	10^{-4}	0.759	54
4	10^{-2}	0.496	4
4	8×10^{-4}	0.569	12

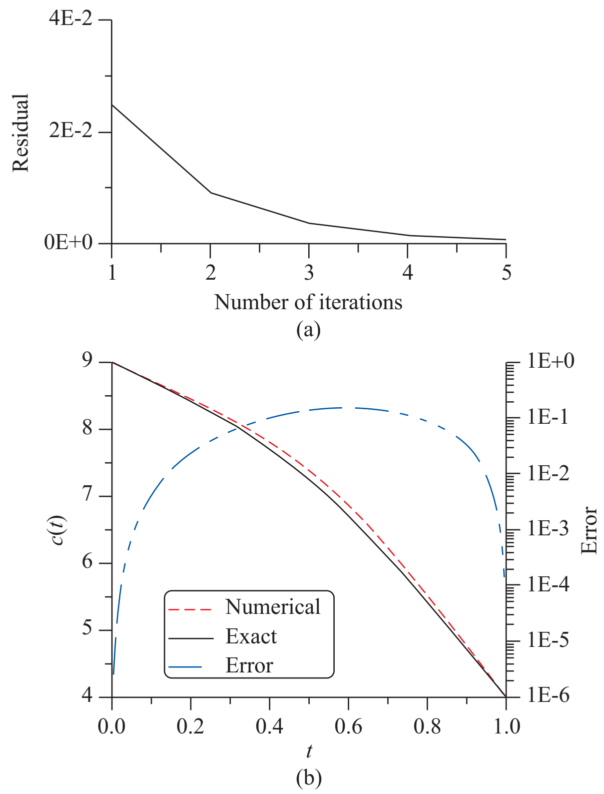

Fig. 1. For the second-order nonlinear oscillator of example 1 solved by the ETBFM iterative algorithm, (a) convergence rate, and (b) comparing recovered and exact damping coefficients.

and iterative numbers increase when using the stringent critical condition and large m.

Example 2

This example is given by

$$c(t) = 3t^2 + 2t^3 + \exp(t^2 + 2t - 1),\ H(x) = x - 10x^2, \quad (25)$$

where $x(t) = \sin(t)$, and the exact $F(t)$ can be derived according to Eq. (1). The above $H(x)$ is the restoring force of a ship rolling nonlinear oscillator (Thompson, 1997).

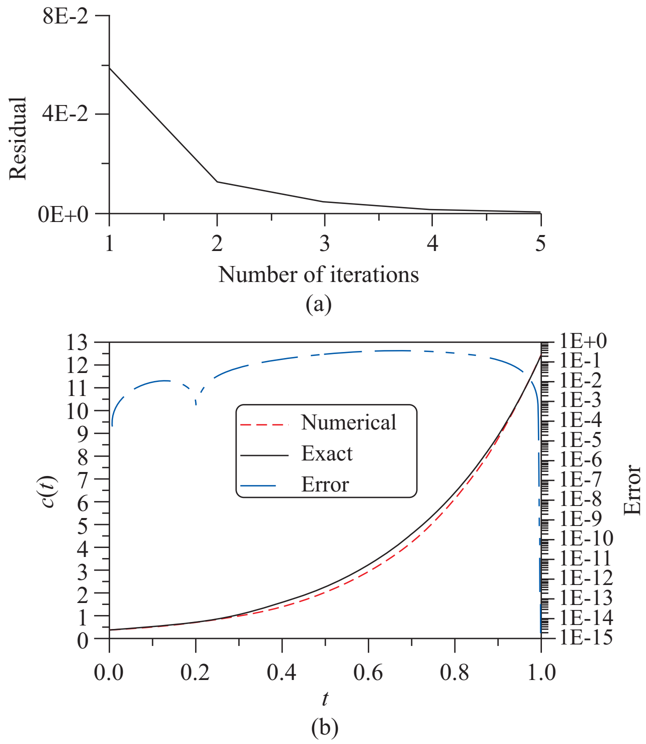

Fig. 2. For the second-order nonlinear oscillator of example 2 solved by the ETBFM iterative algorithm, (a) convergence rate, and (b) comparing recovered and exact damping coefficients.

Under $m = 5$, $a = 1$ and $\varepsilon = 10^{-3}$, the iterative algorithm converges with 5 steps as shown in Fig. 2(a). Upon comparing with the exact $c(t)$, good result is obtained as shown in Fig. 2(b), where the maximum error is 0.355.

IV. IDENTIFYING THE RESTORING FORCES

In this section we attempt to recover the unknown restoring force $H(x)$ through boundary measurements of supplemented data. This problem is more difficult than that in Section 3. In general, the restoring forces problems can be roughly divided into two categories, $H(x, \dot{x})$ and $H(x)$. First, the restoring forces $H(x, \dot{x})$ is a function of both displacement and velocity, which cannot be separable. Chen (2007) and Chen et al. (2013) applied the characteristic time expansion method for estimating nonlinear restoring forces $H(x, \dot{x})$. Another determined restoring forces $H(x)$ with the given damping term, and the restoring forces and damping term can be separable. Previously, Liu (2008b) has developed a Lie-group shooting method to solve this problem, of which however many internal data of displacements are needed.

In order to recover $H(x)$ and avoided multiple roots occurrence, the motion of $x(t)$ is supposed to be a monotonic function of t. We first view $H(t) = H(x(t))$ as a time function of t, and then inverting the relation of $x(t)$ to $t(x)$ and inserting it into $H(t) =$

$H(t(x)) = H(x)$ we can obtain $H(x)$. A monotonic motion is easily obtained by applying a monotonic external force of $F(t)$ on the system.

Here we suppose that $H(t)$ is in the following form:

$$H(t) = D(t) + \sum_{j=1}^{m} c_j E_j(t), \quad (26)$$

where

$$D(t) = \frac{1}{a^3}\left[2H_0 - 2H_a + \dot{H}_0 a + \dot{H}_a a\right]t^3 + \frac{1}{a^2}\left[3H_0 - 3H_a + 2\dot{H}_0 a + \dot{H}_a a\right]t^2 + \dot{H}_0 t + H_0. \quad (27)$$

The data $H_0 = H(0)$, $H_a = H(a)$, $\dot{H}_0 = \dot{H}(0)$ and $\dot{H}_a = \dot{H}(a)$ are supposed to be provided.

The numerical procedures of the energetic temporal boundary function method (ETBFM) to recover H are summarized as follows.

(1) Give m, ε, and an initial guess of $\mathbf{c} = (c_1, \ldots, c_m)^{\mathrm{T}}$,

(2) For $k = 1, \ldots,$ calculate

$$E_j(t) = \gamma_j B_j(t) + B_{j+1}(t),$$

$$H(t) = D(t) + \sum_{j=1}^{m} c_j^k E_j(t),$$

and calculate

$$a_2 = \int_0^a c(t)\, \dot{B}_j^2(t)\, dt,$$

$$a_1 = \int_0^a \left\{2c(t)\left[\dot{B}_{j+1}(t) + \dot{b}(t)\right]\dot{B}_j(t) - \left[H(t) - F(t)\right]\dot{B}_j(t)\right\} dt,$$

$$a_0 = \int_0^a \left\{\begin{array}{l} c(t)\left[\dot{B}_{j+1}(t) + \dot{b}(t)\right]^2 \\ -\left[H(t) - F(t)\right]\left[\dot{B}_{j+1}(t) + \dot{b}(t)\right] \end{array}\right\} dt - D_0,$$

where $D_0 = (\dot{x}_0^2 - \dot{x}_a^2)/2$.

(3) Calculate

$$\gamma_j = \frac{-a_1 - \sqrt{|a_1^2 - 4a_0 a_2|}}{2a_2}$$

$$E_j(t) = \gamma_j B_j(t) + B_{j+1}(t),$$

$$\dot{E}_j(t) = \gamma_j \dot{B}_j(t) + \dot{B}_{j+1}(t),$$

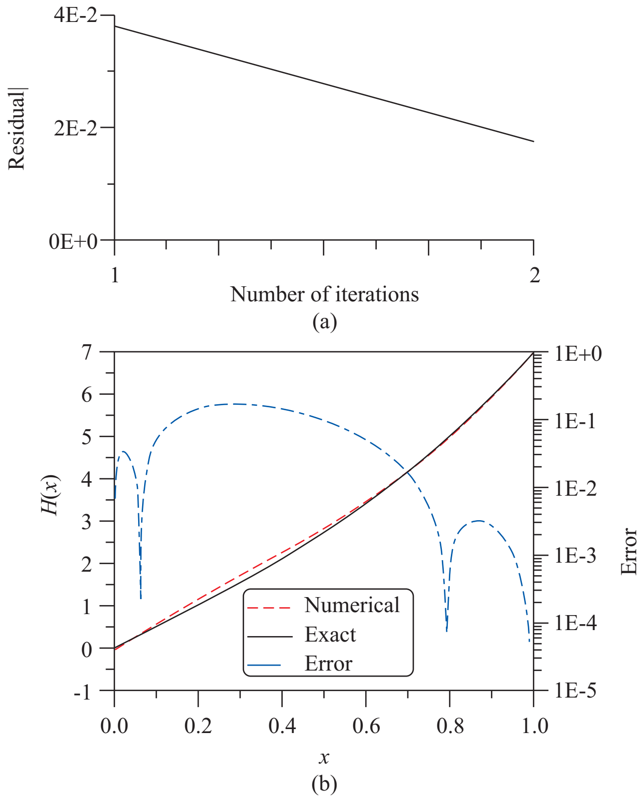

Fig. 3. For the second-order nonlinear oscillator of example 3 solved by the ETBFM iterative algorithm, (a) convergence rate, and (b) comparing recovered and exact restoring force functions.

(4) Insert the above $E_j(t)$ and $\dot{E}_j(t)$ into

$$c_i \int_0^a \left[\dot{E}_j(t) + \dot{b}(t)\right]^2 E_i(t)\, dt =$$

$$D_0 + \int_0^a \left[F(t) - D(t)\right]\left[\dot{E}_j(t) + \dot{b}(t)\right] dt - \int_0^a c(t)\left[\dot{E}_j(t) + \dot{b}(t)\right]^2 dt,$$

and solve this linear system to obtain c_j^{k+1}. If the following convergence criterion for the relative norm of $\mathbf{c}^k$ is satisfied:

$$\left\|\mathbf{c}^{k+1} - \mathbf{c}^k\right\| \le \varepsilon,$$

then stop the iterations; otherwise, go to (2) to the next step. In the first iteration, $a_1^2 - 4a_0 a_2$ may be negative, and we use $|a_1^2 - 4a_0 a_2|$ to avoid the interruption of program.

Example 3

This example is given by

$$H(x) = 5x + 2x^3, \quad (28)$$

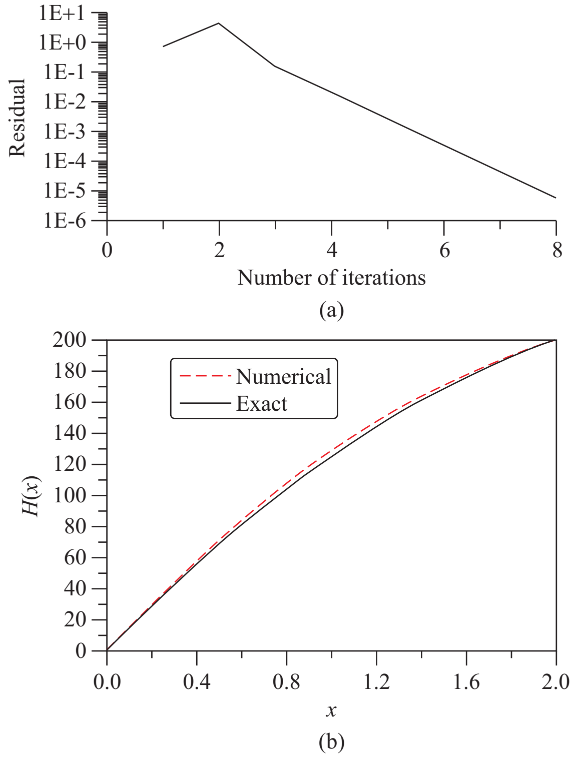

Fig. 4. For the second-order nonlinear oscillator of example 4 solved by the ETBFM iterative algorithm, (a) convergence rate, and (b) comparing recovered and exact restoring force functions.

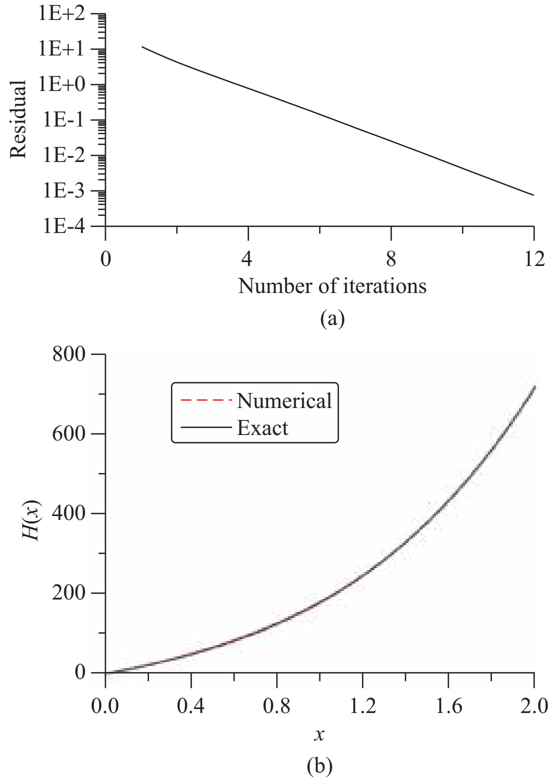

Fig. 5. For the second-order nonlinear oscillator of example 5 solved by the ETBFM iterative algorithm, (a) convergence rate, and (b) comparing recovered and exact restoring force functions.

of a Duffing oscillator. We take $c(t) = \left(t-3\right)^2 + \sin\left(\pi t\right)$, $x(t) = t^2$ and the exact $F(t)$ can be derived according to Eq. (1). Under m = 2, a = 1 and $\varepsilon = 10^{-2}$, the iterative algorithm converges with 2 steps as shown in Fig. 3(a). Upon comparing with the exact $H(x)$, good result is obtained with the maximum error being 0.173 as shown in Fig. 3(b).

Example 4

In this example we recover

$$H(x) = 150x + 25x^2, \tag{29}$$

of a ship rolling oscillator; we take $c(t) = \left(t-3\right)^2 + \sin\left(\pi t\right)$, $x(t) = 2t$ and the exact $F(t)$ can be derived according to Eq. (1). Under m = 1, a = 1 and $\varepsilon = 10^{-5}$, the iterative algorithm converges with 8 steps as shown in Fig. 4(a). Upon comparing with the exact $H(x)$, good result is obtained with the maximum error being 3.77 as shown in Fig. 4(b).

Example 5

The restoring force to be recovered is

$$H(x) = 100x + 50x^2 + 20x^3 + 10x^4. \tag{30}$$

We take $c(t) = \left(t-3\right)^2 + \sin\left(\pi t\right)$, $x(t) = 2t$ and the exact $F(t)$ can be derived by Eq. (1). Under m = 2, a = 1 and $\varepsilon = 10^{-3}$, the iterative algorithm converges within 12 steps as shown in Fig. 5(a). Upon comparing with the exact $H(x)$, good result is obtained with the maximum error being 2.47 as shown in Fig. 5(b).

Example 6

The restoring force to be recovered is

$$H(x) = 300x - 30e^x, \tag{31}$$

and we take $c(t) = \left(t-3\right)^2 + \sin\left(\pi t\right)$, $x(t) = 2t$ and the exact $F(t)$ can be derived according to Eq. (1). Under m = 2 and $\varepsilon = 10^{-3}$, the iterative algorithm converges within 35 steps as shown in Fig. 6(a). Upon comparing with the exact $H(x)$, whose maximum value is about 391, reasonable result is obtained with the maximum error being 28.91 shown in Fig. 6(b).

V. CONCLUSIONS

In this paper we have transformed the dynamic inverse problems to recover the damping functions and the restoring forces of the second-order nonlinear oscillators into linear systems to determine the expansion coefficients of the unknown functions.

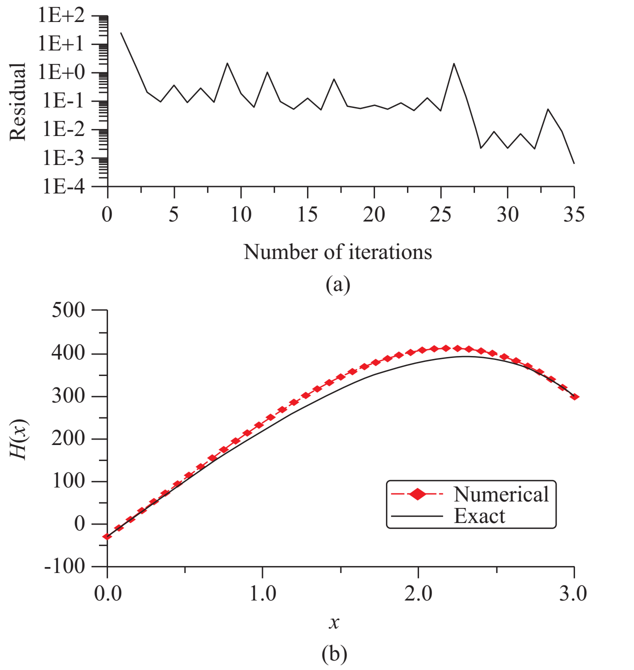

Fig. 6. For the second-order nonlinear oscillator of example 6 solved by the ETBFM iterative algorithm, (a) convergence rate, and (b) comparing recovered and exact restoring force functions.

The energy identity derived from the oscillatory equation was used to set up a linear space of the energetic temporal boundary functions, which not only satisfy the given temporal boundary conditions but also preserve the energy. We can quickly recover the unknown damping functions and the nonlinear oscillators in the linear space, which is supplemented by extra temporal boundary data. Six numerical examples are used to confirm the efficiency and accuracy of the presented energetic temporal boundary functions method, of which the convergence is very fast, from two steps to thirty-five steps.

ACKNOWLEDGEMENTS

The Thousand Talents Plan of China under the Grant Number A1211010 for the financial support to the first author is highly appreciated. Since 2017, CS is a Chair Professor of National Taiwan Ocean University, which is acknowledged. Also, the second author appreciates the financial support provided by the Ministry of Science and Technology, Taiwan, ROC under contract number MOST 107-2221-E-019-029.

REFERENCES

Adhikari, S. and J. Woodhouse (2001a). Identification of damping: part 1, viscous damping. Journal of Sound and Vibration 243, 43-61.

Adhikari, S. and J. Woodhouse (2001b). Identification of damping: part 2, viscous damping. Journal of Sound and Vibration 243, 63-88.

Chen, Y. W. (2007). Application of the characteristic time expansion method for estimating nonlinear restoring forces. Journal of Applied Mathematics 2013, ID 841690, 13.

Chen, Y. W., J. R. Chang, F. H. Hsieh and C. W. Chen (2013). Application of the multi scaling characteristic time expansion method for estimating nonlinear restoring forces. Structural Longevity 9, 37-58.

Crawley, E. F. and A. C. Aubert (1986). Identification of nonlinear structural elements by force-state mapping. AIAA Journal 24, 155-162.

Crawley, E. F. and K. J. O′Donnell (1986). Identification of nonlinear system parameters in joints using the force-state mapping technique. AIAA Paper 86-1013, 659-667.

Duym, S., J. Schoukens and P. Guillaume (1995). A local restoring force surface method. Proceedings of the thirteen International Modal Analysis Conference, Nashville, Tennessee, 1392-1399.

Hasanov, A. and T. S. Shores (1997). Solution of an inverse coefficient problem for an ordinary differential equation. Applicable Analysis 67, 11-20.

Hasanov, A. and B. Pektas (2002). Simulation of ill-conditioned situations in inverse coefficient problem for the Sturm-Liouville operator based on boundary measurements. Mathematics and Computers in Simulation 61, 47-52.

Hasanov, A. and Z. Seyidmamedov (2002). Determination of leading coefficient in Sturm-Liouville operator from boundary measurements. II. unicity and an engineering approach. Applied Mathematics and Computation 125, 23-34.

He, J., B. Xu and S. F. Masri (2012). Restoring force and dynamic loadings identification for a nonlinear chain-like structure with partially unknown excitations. Nonlinear Dynamics 69, 231-245.

Huang, C. H. (2001). A non-linear inverse vibration problem of estimating the time-dependent stiffness coefficients by conjugate gradient method. International Journal for Numerical Methods in Engineering 50, 1545-1558.

Ingman, D. and Z. Suzdalnitsky (2001). Iteration method for equation of viscoelastic motion with fractional differential operator of damping. Computer Methods in Applied Mechanics and Engineering 190, 5027-5036.

Kerschen, G., K. Worden, F. Vakakis and J. C. Golinval (2006). Past, present and future of nonlinear system identification in structural dynamics. Mechanical Systems and Signal Processing 20, 505-592.

Liang, J. W. and B. F. Feeny (2006). Balancing energy to estimate damping parameters in forced oscillators. Journal of Sound and Vibration 295, 988-998.

Liang, J. W. and B. F. Feeny (2011). Balancing energy to estimate damping parameters in forced oscillator with compliant contact. Journal of Sound and Vibration 330, 2049-2061.

Liu, C.-S. (2008a). A Lie-group shooting method for simultaneously estimating the time-dependent damping and stiffness coefficients. Computer Modeling in Engineering and Sciences 27,137-149.

Liu, C.-S. (2008b). A Lie-group shooting method estimating nonlinear restoring force in a mechanical system. Computer Modeling in Engineering and Sciences 35,157-180.

Liu, C.-S. (2011). An inverse problem for computing a leading coefficient in the Sturm-Liouville operator by using the boundary data. Applied Mathematics and Computation 218, 4245-4259.

Liu, C.-S. and J. R. Chang (2016). A real-time Lie-group differential algebraic equations method to solve the inverse nonlinear vibration problems. Inverse Problems in Science and Engineering 24, 1569-1586.

Liu, C.-S., J. R. Chang and Y. W. Chen (2016). The recovery of external force in nonlinear system by using a weak-form integral method. Nonlinear Dynamics 86, 987-998.

Masri, S. F. and T. K. Caughey (1979). A nonparametric identification technique for nonlinear dynamic problems. Journal of Applied Mechanics 46, 433-447.

Namdeo, V. and C. S. Manohar (2008). Force state maps using reproducing kernel particle method and kriging based functional representations. Computer Modeling in Engineering and Sciences 32, 123-160.

Seyidmamedov, Z. and A. Hasanov(2002). Determination of leading coefficient in Sturm-Liouville operator from boundary measurements. I. a stripping algorithm. Applied Mathematics and Computation 125, 1-21.

Thompson, J. M. T. (1997). Designing against capsize in beam seas: recent advance and new insights. ASME Applied Mechanics Reviews 50, 307-325.

Xu, B., J. He and S. F. Masri (2012). Data-based identification of nonlinear restoring force under spatially incomplete excitations with power series polynomial model. Nonlinear Dynamics 67, 2063-2080.