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CONTROL BARRIER FUNCTION DESIGN USING REVIVED TRANSFORMATION

Maki Takai, Motoi Igarashi and Hisakazu Nakamura

Key words: nonlinear control, state constraint, revived transformation.

ABSTRACT

The state constraint problem is an essential topic in control theory, wherein control methods using a control barrier function (CBF) and a revived transformation have been recently proposed. However, a way of designing the CBF is yet to be developed in a constrained space. In this study, we propose a CBF design method by using a revived transformation. Our method can design a CBF in a constrained space by utilizing a diffeomorphism from an unconstrained space. We demonstrate the effectiveness of the proposed method by human assist control design and computer simulation.

I. INTRODUCTION

Considering state constraints is essential in control theory. For example, the movable field in mobile robot control and the rated current in motor control are typical state constraints.

One of the design tools for the state constraint problem involves using a control Lyapunov function (CLF). Control law designs with a CLF are often used to design feedback controllers for nonlinear systems. More recently, several control methods have been proposed that use control barrier functions (CBFs) for the state constraint problems (Ames et al., 2017; Takano and Yamakita, 2018; Zheng et al. 2019; Kolathaya and Ames, 2019; Srinivasan et al., 2018; Wieland and Allgower, 2007; Romdlony and Jayawardhana, 2016). The CBF can be considered as an extension to the definition of the CLF, and a control law combining the CLF and the CBF has been proposed (Mahony and Jiang, 2005; Tee et al., 2009; Romdlony and Jayawardhana, 2014). The above studies mainly consider stabilization problems. However, they do not discuss human assist control. Nakamura et al. (2019) proposed a relaxed CBF based on the CBF proposed by Ames et al. (2015, 2017) and that of Wieland et al. (2007). Moreover, a safety assist control that uses a CBF was proposed (Nakamura et al., 2019). However, CBF designs are generally difficult; to the best of our knowledge, a method of designing the CBF has yet to be established (Xiao and Belta, 2019).

Paper submitted 01/20/20; revised 04/19/20; accepted 05/04/20. Author for correspondence: Hisakazu Nakamura (e-mail: nakamura@rs.tus.ac.jp).
Department of Electrical Engineering, Tokyo University of Science, Chiba, County, Japan

In this paper, we propose a method of designing a CBF by using the revived transformation proposed by Kimura et al. (2015). The revived transformation is a control technique for the state constrained control problem and involves designing coordinates and inputs transformations.

We demonstrate that the CBF property in a constrained space holds under a diffeomorphic coordinate transformation if the CBF in the unconstrained space is a proper function. Thus, if the revived transformation is given, we can design the CBF in the constrained space. We confirmed the effectiveness of the proposed method using an example. Finally, we compared the CBF in Nakamura et al. (2019) to the CBF using the proposed method. The results demonstrate the superiority of the proposed method.

II. PRELIMINARIES

In this section, we introduce basic definitions of mathematical notations, terms, and their fundamental properties used in the paper.

1. Revived system transformation (Kimura et al., 2015)

We consider the following control system:

$$\dot{x} = f(x) + g(x)(u + u_h), \tag{1}$$

where $x \in X = \mathbb{R}^n$ is a state, $u \in \mathbb{R}^m$ is an input, u_h: $\mathbb{R} \to \mathbb{R}^m$ is a human input, and f: $\mathbb{R}^n \to \mathbb{R}^n$ and g: $\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth mappings. Moreover, the state space X denotes a constrained space; that is, $x \in X$ implies that the state constraint is satisfied, otherwise $x \in \mathbb{R}^n \backslash X$ violates the state constraint.

Consider a coordinate transformation ϕ: $\mathbb{R}^n \to X$. Then, the virtual system of (1) on X becomes as follows:

$$\dot{\xi} = \left[\frac{\partial \phi}{\partial \xi}(\xi)\right]^{-1} \left[f\left(\phi(\xi)\right) + g\left(\phi(\xi)\right)\left(u + u_h\right)\right], \tag{2}$$

where $\xi = \phi^{-1}(x) \in \mathbb{R}^n$. The revived transformation is then defined as follows by using an input transformation ψ: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$.

Definition 1 (Revived system transformation) Consider (1), a C^1 diffeomorphism ϕ: $\mathbb{R}^n \to X$, and an input transformation ψ: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Then, a pair of mappings (ϕ, ψ) of ϕ: $\Xi \to X$; $\xi \mapsto x$, ψ: $\Xi \times \mathbb{R}^m \to \mathbb{R}^m$; $(\xi, v, \mathrm{v_h}) \mapsto (u, \mathrm{u_h})$ is said to be a revived system transformation if a virtual system holds the following:

$$\begin{aligned}\dot{\xi} &= \left[\frac{\partial\phi}{\partial\xi}(\xi)\right]^{-1}\left[f(\phi(\xi)) + \mathrm{g}(\phi(\xi))(\psi(\xi, v, v_h))\right] \\ &= f(\xi) + \mathrm{g}(\xi)(v + v_h),\end{aligned} \tag{3}$$

where $\xi \in \mathbb{R}^n$ is a virtual state, $v \in \mathbb{R}^m$ is a virtual control input, and v_h: $\mathbb{R} \to \mathbb{R}^m$ is a virtual human input.

To design a revived transformation, we assume the following conditions.

Assumption 1 $g(x)$ in (1) has full column rank for all $x \in X$, and matrix $G(x)$ defined by the following, is non-singular for all $x \in X$:

$$G(x) = \begin{bmatrix} g^+(x) \\ g^\perp(x) \end{bmatrix}, \tag{4}$$

where $g^+(x)$ is a left inverse, and $g^\perp(x)$ is a left zero divisor for all $x \in X$.

Assumption 2 Diffeomorphism ϕ: $\Xi \to X$; $\xi \mapsto x$ satisfies the following conditions:

$$g^\perp(x) \cdot \frac{\partial\phi}{\partial\xi} \cdot g(\xi) = O_{(n-m,m)}, \tag{5}$$

$$g^\perp(x) \cdot \frac{\partial\phi}{\partial\xi} \cdot f(\xi) = g^\perp(x) \cdot f(\phi(\xi)), \tag{6}$$

Under Assumption 1 and 2, the following theorem holds.

Theorem 1 Consider system (1) and suppose Assumptions 1 and 2.

Then, a pair of mappings (ϕ, ψ) is a revived transformation, where mapping ψ: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is defined as follows:

$$\begin{aligned}u = \psi(\xi, v, v_h) &= g^+(\phi(\xi)) \cdot \frac{\partial\phi}{\partial\xi} \cdot \{f(\xi) + g(\xi)(v + v_h)\} \\ &-g^+(\phi(\xi))f(\phi(\xi)) - u_h.\end{aligned} \tag{7}$$

Proof. Under Assumption 2, note that the following equalities hold (Kimura et al., 2015).

$$\left[\frac{\partial\phi}{\partial\xi}(\xi)\right]^{-1} g(\phi(\xi))g^+(\phi(\xi))\frac{\partial\phi}{\partial\xi}g(\xi) = g(\xi), \tag{8}$$

$$\begin{aligned}&\left[\frac{\partial\phi}{\partial\xi}(\xi)\right]^{-1} [f(\phi(\xi)) + g(\phi(\xi))g^+(\phi(\xi))\frac{\partial\phi}{\partial\xi}f(\xi) \\ &-g(\phi(\xi))g^+(\phi(\xi))f(\phi(\xi)) = f(\xi).\end{aligned} \tag{9}$$

Then substituting (8) and (9) into (2), we obtain the following equation:

$$\dot{\xi} = f(\xi) + g(\xi)(v + v_h). \tag{10}$$

2. Control Lyapunov function (CLF) (Artstein, 1983)

In this paper, we propose a CBF design method by using a CLF defined as follows.

Definition 2 (Control Lyapunov function (CLF)) We consider (1) and a C^1 continuously differentiable function V: $\mathbb{R}^n \to \mathbb{R} \geq 0$. It is said to be a CLF if the following three conditions hold.

(A1) Non-negative function; $V(x) \geq 0$ for all $x \in X$.
(A2) Proper function; $\{x \mid V(x) \leq L\}$ is compact for any $L \in \mathbb{R}$.
(A3) The following equation holds:

$$\begin{aligned}&\inf_{u\in\mathbb{R}^m} \dot{V}(x, u_h(t), u) \\ &= \inf_{u\in\mathbb{R}^m} [L_f V(x) + L_g V(x)(u + u_h(t))] < 0,\end{aligned} \tag{11}$$

where $L_f V = \partial V/\partial x \cdot f$ and $L_g V = \partial V/\partial x \cdot g$.

3. Control barrier function by Ames et al. (2014, 2017)

Ames et al. proposed a reciprocal CBF, which has a different framework from the one proposed by Wieland et al.

In the reciprocal CBF, safe set $X \subset \mathbb{R}^n$ is supposed to be a closed set. Moreover, Int(X) denotes the interior (without boundary) of set X, ∂X the boundary of X, respectively; we suppose that X is not empty and that it does not consist of only a single element.

To define the reciprocal CBF, we need to define a function h: $X \to \mathbb{R}$, such that the following three conditions hold:

$$\begin{aligned}&h(x) \geq 0 \forall x \in X \\ &h(x) = 0 \forall x \in \partial X \\ &h(x) > 0 \forall x \in \mathrm{Int}(X).\end{aligned} \tag{12}$$

Then, the reciprocal CBF is defined as follows:

Definition 3 (Reciprocal Control Barrier Function) Consider system (1), a safe closed-set $X \subset \mathbb{R}^n$, and a C^1 differentiable function $h(x)$ that satisfies (12). Then, a C^1 differentiable function B: Int(X) $\to \mathbb{R}$ is said to be a reciprocal CBF if there are three class K functions α_1, α_2 and α_3 such that the following two inequalities hold:

$$\frac{1}{\alpha_1(h(x))} \le B(x) \le \frac{1}{\alpha_2(h(x))}, \tag{13}$$

$$\inf_{u\in\mathbb{R}^m}[L_f B(x) + L_g B(x)(u + u_h(t)) - \alpha_3(h(x))] \le 0. \tag{14}$$

Note that α_3 must be a class K function and $h(x) = 0$ for all $x \in \partial X$ due to the definition of a reciprocal CBF. Then, (11) uniformly converges to the following inequality as a $x \to \partial X$:

$$\inf_{u\in\mathbb{R}^m}[L_f B(x) + L_g B(x)u] \le 0. \tag{15}$$

4. Control barrier function by Nakamura et al. (2019)

Definition 4 (Control Barrier Function) Consider (1). A C^1 continuously differentiable function $B: X \to \mathbb{R}$ is said to be a CBF if the following three conditions hold.

(B1) Non-negative function; $B(x) \ge 0$ for all $x \in X$.
(B2) Proper function; $\{x \mid B(x) \le L\}$ is compact for any $L \in \mathbb{R}$.
(B3) For any continuous mapping $u_h: \mathbb{R} \to \mathbb{R}^m$, there exist constants $C, K > 0$ such that the following inequality holds:

$$\begin{aligned}&\inf_{u\in\mathbb{R}^m} \dot{B}(x, u_h(t), u)\\ &= \inf_{u\in\mathbb{R}^m}[L_f B(x) + L_g B(x)(u + u_h(t))]\\ &< KB(x) + C.\end{aligned} \tag{16}$$

For every relaxed CBF, B must be infinity on a boundary of the safe set. More precisely, for any convergence sequence $(x_i)_{i\in N}$ such that $x_i \to \partial X$ as $i \to \infty$, $B(x_i) \to \infty$ as $i \to \infty$.

5. Safety assist control by using a CBF (Nakamura et al., 2019)

The following theorem holds by using a CBF:

Theorem 2 We consider (1) and a CBF $B: X \to \mathbb{R}$ satisfying condition (B3).

Then, $x(t) \in X$ for all $t \ge 0$ and any continuous mapping u_h: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ by the following control input $u = k(x, t)$,

$$u = k(x,t) = \begin{cases} 0 & (I \le J) \\ -\dfrac{I(x,u_h) - J(x)}{\left\|L_g B(x)\right\|^2}\left(L_g B(x)\right)^T & (I > J) \end{cases}, \tag{17}$$

where functions I: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and J: $\mathbb{R}^n \to \mathbb{R}$ are defined by the following equations, respectively.

$$\begin{aligned} I(x,u_h) &= L_f B(x) + L_g B(x)\cdot u_h(t),\\ J(x) &= KB(x) + C. \end{aligned} \tag{18}$$

Proof. Note that $L_f B(x)$, $L_g B(x)$ and $u_h(t)$ are all continuous mappings, and functions I and J are continuous. Moreover, $u \to 0$ as $I \to J$ uniformly when $L_g B(x) \ne 0$. Hence, (17) is continuous if $L_g B(x) \ne 0$. Then, we prove the continuity of (17) on $L_g B(x) = 0$.

According to (B3) in the definition of CBF, for all $t_0 \ge 0$ such that $L_g B(x(t_0)) = 0$, the following inequality holds:

$$\dot{B} = L_f B(x(t_0)) < KB(x(t_0)) + C. \tag{19}$$

This implies that there exists $\varepsilon > 0$ such that the following inequality holds:

$$\dot{B} = L_f B(x(t_0)) + \varepsilon \le KB(x(t_0)) + C. \tag{20}$$

Since $L_g B(x) u_h(t)$ is a continuous mapping, there exists a neighborhood $M \subset \mathbb{R}^n \times \mathbb{R}$ of $x_0 = x(t_0)$ and t_0 such that the following inequality holds:

$$L_g B(x) u_h(t) \le \varepsilon \quad \forall (x,t) \in M. \tag{21}$$

Hence the following inequality holds for any $(x, t) \in M$:

$$L_f B(x) + L_g B(x) u_h(t) \le KB(x) + C. \tag{22}$$

Then, the following inequality holds:

$$I(x, u_h) \le J(x). \tag{23}$$

Therefore, $k(x, t) = 0$ for any (x, t) in set M that is a neighborhood of (x_0, t_0).This implies that there exists a neighborhood M of $L_g B(x) = 0$ such that $k(x, t) = 0$ for all $(x, t) \in M$. Accordingly, $k(x, t)$ is continuous on the set $\{x \mid L_g B(x) = 0\}$. Therefore, k: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is a continuous mapping, and there exists a local solution of differential equation (1) with (17).

Consider (1) and (17). If $I \le J$, the following inequality holds:

$$L_f B(x) + L_g B(x) u_h(t) = I(x, u_h) \le J(x) = KB(x) + \mathrm{C}. \tag{24}$$

On the other case that $I > J$, the following inequality holds:

$$L_f B(x) + L_g B(x) u_h(t) - (I - J) = J = KB + C. \tag{25}$$

Hence by (24) and (25), the following inequality holds for all $(x, t) \in X \times \mathbb{R}$:

$$\dot{B} \le KB + C. \tag{26}$$

According to Gronwall's inequality (Cannarsa, 2004), the following inequality holds:

$$B(t) \le B(0)e^{Kt} + \frac{C}{K}e^{Kt} - \frac{C}{K}. \tag{27}$$

Therefore for any $t > 0$, $B(x(t)) \in \mathbb{R}$. This implies $x(t) \in X$ owing to (B2).

"<" in (16) in the definition of a CBF cannot be replaced by "≤" for the continuity of a human assist input (17), that is, (17) may be discontinuous at $I = J$ in the case of "≤."

III. PROBLEM STATEMENT

In this paper, we consider system (1) as a control system. Moreover, we divide the domain of x into two subsets: a safe set $X \subset \mathbb{R}^n$, and an unsafe set $X_u = \mathbb{R}^n \backslash X \subset \mathbb{R}^n$. If $x \in X$, the system is said to be safe. Otherwise if $x \in X_u$, the system is said to be unsafe. We suppose that X is diffeomorphic to $\mathbb{R}^n$, and a diffeomorphism ϕ: $\mathbb{R}^n \to X$ is given.

The objective of the paper is to design the CBF on X.

IV. INVARIANCE OF CONTROL BARRIER FUNCTION

We here propose our design strategy of a CBF on X. We show that $B(\phi^{-1}(x))$ is the CBF on X, and that a property of the CBF is invariant when the revived transformation (ϕ, ψ) is given for system (1). In addition, we show the CLF, as a property of a CBF, becomes the CBF itself.

Proposition 1 We consider (3) and a CBF $B(\xi)$. We assume the revived transformation (ϕ, ψ) is given. Then, $B(\phi(x))$ on X is a CBF for (1) that satisfies conditions (B1), (B2), (B3) in Definition 4.

Proof. We consider (1) and $B(\xi)$ on $\mathbb{R}^n$. Note that X is diffeomorphic to $\mathbb{R}^n$, and $x = \phi(\xi(t)) \in X$ holds. Thus, $B(\phi(x))$ satisfies the conditions (B1) and (B2) in Definition 4.

Moreover, owing to the property of the revived transformation (ϕ, ψ), (1) is transformed to (3). $\dot{B}(\xi)$ holds the following equation:

$$\begin{aligned} \dot{B}(\xi) &= \frac{\partial B}{\partial \xi} \cdot \dot{\xi} \\ &= \frac{\partial B}{\partial \xi}\left[\left(\frac{\partial \phi}{\partial \xi}\right)^{-1}\left\{f(\phi(\xi)) + g(\phi(\xi))(v + v_h(t))\right\}\right]. \end{aligned} \tag{28}$$

Then, the following inequality holds by condition (B3) in Definition 4:

$$\dot{B}(\xi) < KB(\xi) + C. \tag{29}$$

Hence, the following equation holds:

$$\begin{aligned} \dot{B}(\phi^{-1}(x)) &= \frac{\partial B}{\partial \xi} \cdot \left(\frac{\partial \phi}{\partial \xi}\right)^{-1} \dot{x} \\ &= \frac{\partial B}{\partial \xi} \cdot \left(\frac{\partial \phi}{\partial \xi}\right)^{-1} \cdot \left(\frac{\partial \phi}{\partial \xi}\right) \cdot \dot{\xi} \\ &= \frac{\partial B}{\partial \xi} \cdot \dot{\xi}. \end{aligned} \tag{30}$$

According to (28) and (29), the following inequality holds:

$$\dot{B}(\phi^{-1}(x)) < KB(\phi^{-1}(x)) + C. \tag{31}$$

Therefore, $B(\phi(x))$ is a CBF that satisfies conditions (B1), (B2), and (B3) in Definition 4.

Note that in the virtual space, the state constraint appears to have disappeared. This fact reveals the following corollary of Proposition 1:

Corollary 1 We consider (3) and a CLF $V(\xi)$. Then, $V(\phi^{-1}(x))$ is a CBF.

Proof. According to Definitions 2 and 4, conditions (A1) and (A2) are the same as (B1) and (B2). Moreover, the following inequality holds for all K, $C > 0$:

$$\inf_{v \in \mathbb{R}^m} \frac{\partial V}{\partial \xi} \dot{\xi} < 0 < KV(\xi) + C. \tag{32}$$

The rest of the proof is the same as for Proposition 1.

According to the same discussion as that given for Corollary 1, we can show that a function with weaker requirements than those for the CLF defined by Definition 2 becomes a CBF:

Corollary 2 We consider that (3) and $V'(\xi)$ satisfy the conditions (A1) and (A2) and the following condition (A3').

(A3') The following inequality holds:

$$\begin{aligned} &\inf_{v \in \mathbb{R}^m} \dot{V}'(\xi, v_h(t), v) \\ &= \inf_{v \in \mathbb{R}^m} \left[L_f V'(\xi) + L_g V'(\xi)(v + v_h(t))\right] \le 0, \end{aligned} \tag{33}$$

Then, $V'(\phi^{-1}(x))$ is a CBF.

Proof. According to Definitions 2 and 4, conditions (A1) and (A2) are the same as (B1) and (B2). Moreover, the following inequality holds for all K, $C > 0$:

$$\inf_{v \in \mathbb{R}^m} \frac{\partial V'}{\partial \xi} \dot{\xi} \le 0 < KV(\xi) + C. \tag{34}$$

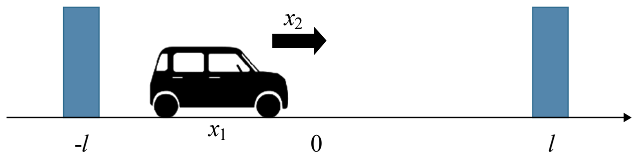

Fig. 1. Control system: example

Remark 1 Ames et al. did not discuss the need for proper functions concerning their reciprocal CBF (Ames et al., 2017). If the CBF is not proper, the invariance of the CBF by coordinates transformation will not generally hold.

V. EXAMPLE

In this section, we design a CBF using the proposed method, and we present a safety assist u using the designed CBF $B(\phi^{-1}(x))$.

Firstly, we show that the invariance of the CBF by coordinate transformation does not hold when using a reciprocal CBF.

Secondly, we demonstrate the effectiveness of the proposed method using the proper CBF property.

We consider one-dimensional motion with acceleration input and walls, as illustrated in Fig. 1. The system can be modeled by the following equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_h(t) + u, \end{aligned} \tag{35}$$

where $x_1, x_2 \in X = \mathbb{R}$ denotes a state, and u is an acceleration input. Suppose that the two walls are placed at $-l$ and l, where l is a positive constant and the environment $X = \{x \mid x_1 \in (-l, l), x_2 \in \mathbb{R}\}$.

For system (35), we design the following diffeomorphism ϕ:

$$\phi(\xi) = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2l}{\pi} \tan^{-1}\left(\dfrac{\pi}{2l} \xi_1 \right) \\ \dfrac{\xi_2}{1 + \left(\dfrac{\pi}{2l} \xi_1 \right)^2} \end{bmatrix}. \tag{36}$$

Moreover, the mapping $\phi^{-1}: X \to \Xi$ is obtained as follows:

$$\phi^{-1}(x) = \begin{bmatrix} \phi_1^{-1}(x) \\ \phi_2^{-1}(x) \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2l}{\pi} \tan\left(\dfrac{\pi}{2l} x_1 \right) \\ \dfrac{x_2}{\cos^2\left(\dfrac{\pi}{2l} x_1 \right)} \end{bmatrix}. \tag{37}$$

We consider the system (35) and mapping (36). Then, the following ψ: $(\xi, v\ v_h) \mapsto (u, u_h)$ is an input transformation:

$$\begin{aligned} \psi(\xi, v, v_h) &= u + u_h \\ &= \frac{\pi^2 \xi_1 \xi_2^2}{2l^2 \left\{ 1 + \left(\dfrac{\pi}{2l} \xi_1 \right)^2 \right\}^2} + \frac{1}{1 + \left(\dfrac{\pi}{2l} \xi_1 \right)^2} (v + v_h). \end{aligned} \tag{38}$$

Note that the control system (35) on the virtual space is obtained as follows by revived transformation (ϕ, ψ):

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v_h(t) + v. \end{aligned} \tag{39}$$

1. Reciprocal CBF by Ames et al.

We consider a reciprocal CBF by Ames et al. on $\mathbb{R}^2$ as follows:

$$B(\xi) = 0, \tag{40}$$

According to (37), we can obtain a function $B(\phi^{-1}(\xi))$ on X as follows:

$$B(\phi^{-1}(x)) = 0. \tag{41}$$

Then, we can permit that (41) does not satisfy the definition of a reciprocal CBF on X.

2. The CBF by Nakamura et al. and human assist control

We consider a CBF on $\mathbb{R}^2$ as follows:

$$B(\xi) = \xi_1^2 + \xi_2^2. \tag{42}$$

The following proposition holds for (42):

Proposition 2(42) satisfies the conditions (B1), (B2) and (B3) in Definition 4.

Proof. Firstly, we prove that (42) satisfies the condition (B1). Further, (42) holds the following equation:

$$B(\xi) = \xi_1^2 + \xi_2^2 > 0, \tag{43}$$

where (42) is a non-negative function. Thus, (42) satisfies the condition (B1). Then, we prove that (42) satisfies the condition (B2). For $\xi \in \mathbb{R}^n$, we consider the following inequality:

$$\xi_1^2 + \xi_2^2 \le L. \tag{44}$$

Then, (42) is obviously bounded. Moreover, we consider the following set Y:

$$Y = \{\xi \mid \xi_1^2 + \xi_2^2 \le L\}, \tag{45}$$

when the complementary set Y^c is :

$$Y^c = \{\xi \mid \xi_1^2 + \xi_2^2 > L\}. \tag{46}$$

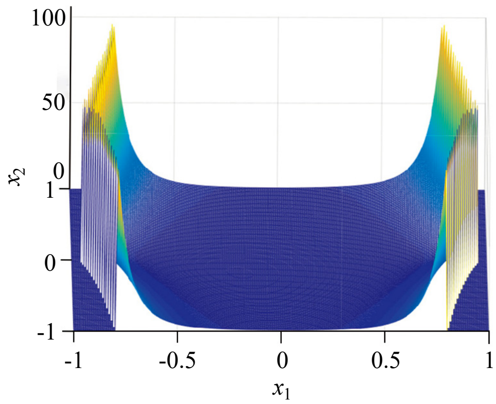

Fig. 2. Designed CBF $B(\phi^{-1}(x))$

Here, Y^c is clearly an open set. Thus, (45) is a closed set. Therefore, (42) satisfies the condition (B2).

Finally, we prove that (42) satisfies the condition (B3). The time derivative of the CBF is calculated as follows:

$$\dot{B} = \frac{\partial B}{\partial \xi}\frac{\partial \xi}{\partial t} = L_f B(\xi) + L_g B(\xi)\cdot(u + u_h). \quad (47)$$

If $u = -u_h - L_f B / L_g B$, $\dot{B} = 0$. Thus,

$$\dot{B} = 0 < KB + C. \quad (48)$$

Therefore, (42) satisfies the condition (B3).

According to (37), we can obtain the CBF $B(\phi^{-1}(x))$ on X as follows:

$$B(\phi^{-1}(x)) = \left\{\frac{2l}{\pi}\tan\left(\frac{\pi}{2l}x_1\right)\right\}^2 + \left\{\frac{x_2}{\cos^2\left(\frac{\pi}{2l}x_1\right)}\right\}^2. \quad (49)$$

(49) is illustrated in Fig. 2. According to Proposition 1, (49) is a CBF that satisfies the conditions (B1), (B2), and (B3) in Definition 4. From the above, it is important to consider the property of the condition (B2) in the proposed method.

Hence, a safety assist control u was designed by using (49) as follows:

$$u = k(x,t) = \begin{cases} 0 & (I \le J) \\ -\dfrac{I(x,u_h) - J(x)}{\|L_g B(x)\|^2}\left(L_g B(x)\right)^T & (I > J) \end{cases}, \quad (50)$$

where

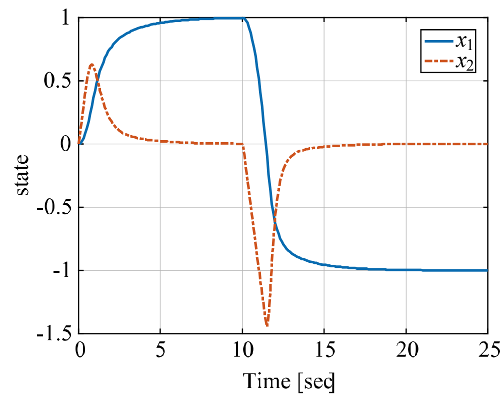

Fig. 3 State by the proposed method (K = 1.0, C = 10.)

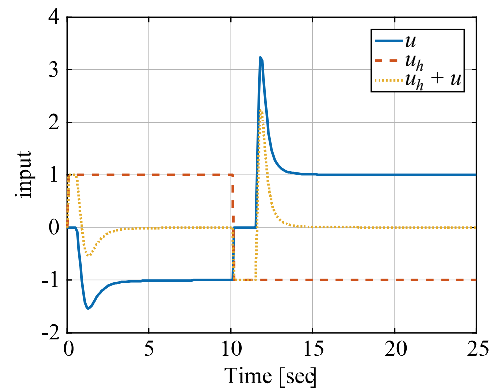

Fig. 4. Input by the proposed method (K = 1.0, C = 1.0)

$$I = \tan\left(\frac{\pi}{2l}x_1\right)\cdot\left(\frac{4l}{\pi}\cdot\frac{x_2}{\cos^2\left(\frac{\pi}{2l}x_1\right)} + \frac{2\pi x_2^3}{\cos^4\left(\frac{\pi}{2l}x_1\right)}\right) + \frac{2x_2}{\cos^4\left(\frac{\pi}{2l}x_1\right)}\cdot u_h \quad (51)$$

$$J = K\left[\left\{\frac{2l}{\pi}\tan\left(\frac{\pi}{2l}x_1\right)\right\}^2 + \left\{\frac{x_2}{\cos^2\left(\frac{\pi}{2l}x_1\right)}\right\}^2\right] + C. \quad (52)$$

3. Computer simulation

In this subsection, we evaluate the effectiveness of the proposed controller (50) with computer simulation.

We set $l = 1$, an initial condition $x_0 = [0, 0]^T$, $K = 1.0$, and $C = 1.0$. Moreover, we set $u_h = 1.0$ in the case of $t < 10$ [s] and $u_h = -1.0$ in the case of $t > 10$ [s].

Figure 3 shows the time response of the state variables, and Fig. 4 shows the time histories of the inputs.

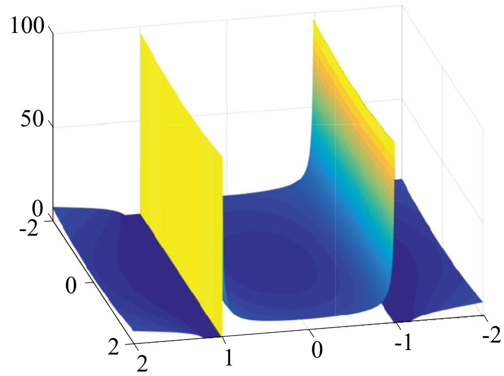

Fig. 5. Simple CBF $B(x)$

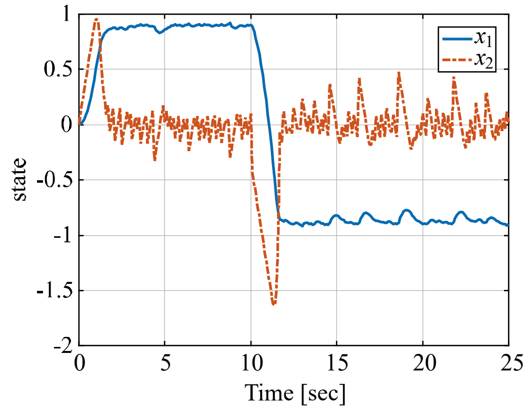

Fig. 6. State by the simple CBF (K = 1.0, C = 1.0)

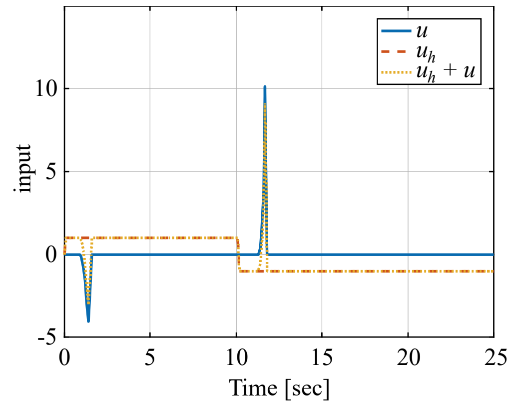

Fig. 7. Input by the simple CBF (K = 1.0, C = 1.0)

From Fig. 3, we can confirm that $-1 < x_1 < 1$ is always satisfied, and that the state stays in the safe set $X = (-1, 1)$. From Fig. 4, we can confirm that $u = 0$ at $t = 0$, $u < 0$ when x_1 is approaching $x_1 = 1$, and $u > 0$ when x_1 is approaching $x_1 = -1$. Thus, we can confirm the effectiveness of the proposed CBF design method using the revived transformation.

VI. COMPARISON

In Nakamura et al. (2019), the CBF was designed as a simple function, by setting the value of the function toward infinity at the boundary of the state constraint. However, sudden assistance was observed, and the parameters K and C in Theorem 2 needed to be appropriately chosen.

In this section, we describe our design of a human assist control using the simple CBF. Moreover, we compare this simple CBF with the one we designed using the proposed method described in the preceding section with computer simulations.

We consider a simple CBF as follows:

$$B(x) = \frac{1}{l - x_1} + \frac{1}{l + x_1} + x_1^2 + x_2^2. \tag{53}$$

(41) is illustrated in Fig. 5. Then, the safety assist control u is designed as follows:

$$u = k(x, t) = \begin{cases} 0 & (I \le J) \\ -\dfrac{I(x, u_h) - J(x)}{\|L_g B(x)\|^2}\left(L_g B(x)\right)^T & (I > J) \end{cases}, \tag{54}$$

$$I = -\frac{x_2}{(x_1 + l)^2} + \frac{x_2}{(l - x_1)^2} + 2x_1 x_2 + 2x_2 \cdot u_h, \tag{55}$$

$$J = K\left(\frac{1}{x_1 + l} + \frac{1}{l - x_1} + x_1^2 + x_2^2\right) + C. \tag{56}$$

1. Computer simulation 1

We show two patterns of computer simulation results using the controllers (50) and (54). We carried out simulations using the Runge-Kutta law with a time step of 0.1.

In this subsection, we use the same conditions as those described in the previous section. Figure 6 shows the time response of the state variables, and Figure 7 shows the time histories of the inputs.

Comparing Fig. 3 with Fig. 6, we find that $-1 < x_1 < 1$ is always satisfied, and the state stays in the safe set $X = (-1, 1)$.

However, from Fig. 6, we can confirm that x_1 does not approach the constraint in comparison with Fig. 3, and x_2 is oscillating. Comparing Fig. 4 with Fig. 7, the assist input u is sharp in Fig. 7. However, we can confirm that the assist input u is moderate without tuning the parameters K and C in Fig. 4.

Moreover, the assist input u is approximately −1.5 around 2 [s], and approximately 3.2 around 12 [s] in Fig. 4. By contrast, the assist input u is approximately −3.4 around 2 [s], and approximately 10 around 12 [s] in Fig. 7. As a result, we can confirm that the proposed method guarantees safety with a smaller assist. The reason is that the proposed method CBF (49) can design the CBF by considering the velocity x_2, unlike the simple CBF (53). We can also confirm te fact from Figs. 2 and 5.

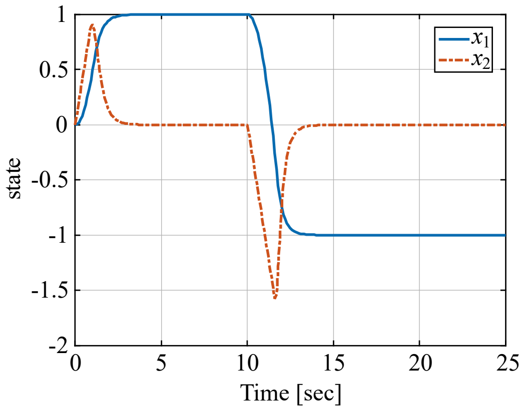

Fig. 8. State by the proposed method (K = 5.0, C = 1.0)

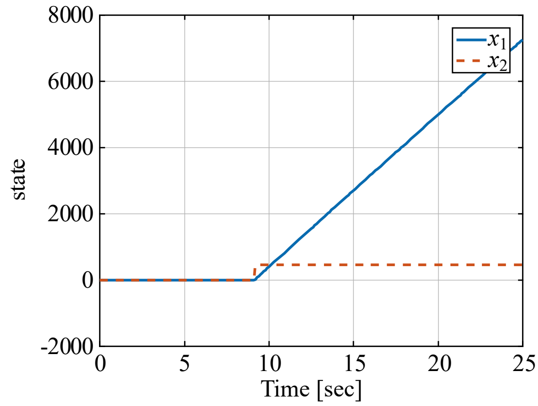

Fig. 10. State by the simple CBF (K = 5.0, C = 1.0)

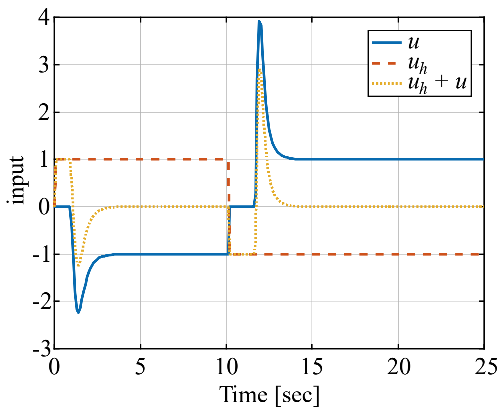

Fig. 9. Input by the proposed method (K = 5.0, C = 1.0)

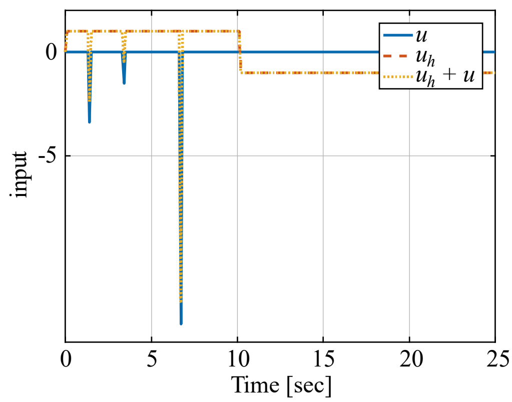

Fig. 11. Input by the simple (K = 5.0, C = 1.0)

2. Computer simulation 2

We here describe another computer simulation. We selected $K = 5.0$ and $C = 1.0$. The other conditions for this computer simulation were the same as those in the preceding subsection.

Figures 8 and 9 depict the simulation results with the proposed method. Figures 10 and 11 show the simulation results for the simple CBF (41).

From Fig. 8, we can confirm that $-1 < x_1 < 1$ is always satisfied. In Fig. 9, we find that the assist input is larger than Fig. 4. However, in Fig. 6, the state constraint $-1 < x_1 < 1$ is not satisfied. As a result, the conventional, simple CBF may not guarantee the state constraint, even at the simulation level. Thus, the proposed method is effective as a design method for CBFs.

VII. CONCLUSION

In this study, we proposed a method of designing a CBF by using a revived transformation. The conditions of the CBF are preserved in constrained space when the proper function that satisfies the condition of the CBF undergoes a diffeomorphic coordinate transformation. We demonstrated that the CBF could be designed when the revived transformation is known. Moreover, the effectiveness of the proposed method was confirmed with an example. Finally, we conducted a comprehensive comparison of the proposed method with a simple CBF.

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