

# CONSTRAINTS TO GUARANTEE GAIN AND PHASE MARGINS FOR DATA-DRIVEN CONTROLLER TUNING METHODS

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Key words: data-driven controller tuning, NCbT, stability constraint, gain and phase margins.

## ABSTRACT

The Noniterative Correlation-based Tuning (NCbT) is one of the data-driven controller tuning methods which directly tune controller parameters from input/output data set of a plant. The NCbT is based on the correlation approach to robustly tune controller parameters using a noisy input/output data set. Since the NCbT guarantees closed-loop stability only for the situation where the data is acquired, a plant fluctuation is not taken into consideration, which may lead to degradation of control performance and/or destabilization of the closed-loop system. In this paper, we virtually produce input/output data sets of the plant with various gain and/or phase fluctuation to derive data-driven constraints for the prespecified gain and phase margins.

## I. INTRODUCTION

Model-based controller tuning methods have received attention in recent years. However, highly complex plants have appeared with the advancement of technologies and equipments, and it is difficult to accurately describe their dynamics with a limited complexity. Under this background, data-driven controller tuning methods are actively studied (Hjalmarsson et al., 1998; Campi et al., 2002; Karimi et al., 2004; Karimi et al., 2007; Saeki, 2014; Date et al., 2018). In these studies, a controller is designed easily by giving the reference model and solving a optimization problem using an input/output data set of a plant instead of using a parametric model. In the Iterative Feedback Tuning method (IFT) (Hjalmarsson et al., 1998) and the Iterative Correlation-based Tuning method (Karimi et al., 2004), a non-convex

optimization problem is treated. These methods require multiple experiments to evaluate a cost function to design a controller. In the Virtual Reference Feedback Tuning (VRFT) (Campi et al., 2002), a convex optimization is treated, which requires just a single experiment to evaluate a cost function. However, this method suffers from the noise on the collected output data when a controller is designed.

In the Noniterative Correlation-based Tuning (NCbT) (Karimi et al., 2007), a convex optimization problem to design a controller is formulated by using a set of input/output data as well as the VRFT. Moreover, the NCbT is insensitive to the noise compared with the VRFT since the correlation approach is adopted. However, if an unsuitable reference model is given, a designed controller may destabilize the closed-loop system. In addition, it is difficult to give analyses of stability and stability margin of the closed-loop system since a plant model is unavailable in the data-driven approaches.

To solve this problem, the methods to check closed-loop stability using input/output data is developed (Sala and Esparza, 2005; Park and Ikeda, 2005). Moreover, the stability constraint based on the small gain theorem is proposed to directly design a stabilizing controller and is incorporated into a design problem (Heusden et al., 2011). This constraint just guarantees stability of the closed-loop system but does not explicitly address robustness issue. Hence, a plant fluctuation might destabilize the closed-loop system if stability margin is not enough ensured. Gain and phase margins are typical stability indices for robustness in classical control theory, which are still used in many fields of applications. In data-driven approach, the explicit stability margins such as gain and phase margins should be incorporated into controller design. The method to guarantee gain and phase margins by using the frequency responses of the open-loop transfer function is developed (Date et al., 2018). However, the design problem in this method is complicated because the constraints for the stability and the stability margin are described in different forms.

For this problem, this paper proposes constraints to guarantee classical stability margins, gain and phase margins, for data-driven controller tuning methods. These constraints are formulated easily using input/output data sets by extending

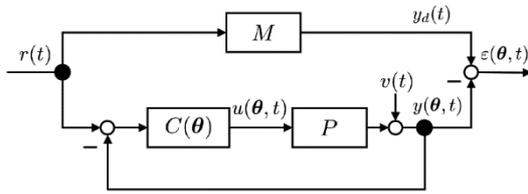


Fig. 1. Model reference control problem

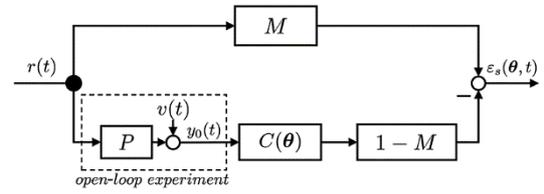


Fig. 2. Equivalent model reference control problem

the existing stability constraint. So, the stability constraint and the stability margin constraints are unified in the same framework. Then, the effectiveness of the proposed method is shown by numerical simulations.

Throughout this paper, a discrete-time transfer function  $P(z^{-1})$  is denoted by  $P$  for simplicity of notations, where  $z^{-1}$  is the backward shift operator.

## II. NONITERATIVE CORRELATION-BASED TUNING

The NCbT proposed by Karimi et al. is one of the data-driven controller tuning methods to solve a model reference control problem. In this section, we review the design problem of the NCbT.

### 1 Problem Setup

A model reference control problem aims to find the controller parameter vector  $\theta$  that minimizes the cost function  $J_{MR}(\theta)$  which is formulated in the frequency domain as

$$J_{MR}(\theta) = \left\| M - \frac{PC(\theta)}{1+PC(\theta)} \right\|_2^2, \quad (1)$$

where  $M$  and  $P$  denote a reference model given by a designer and an unknown linear time-invariant plant, respectively.  $C(\theta)$  is a discrete-time transfer function of the linearly parameterized controller to be designed as

$$C(\theta) = \beta^T \theta, \quad (2)$$

where

$$\theta = [\theta_1, \theta_2, \dots, \theta_{n_c}]^T.$$

$\beta$  is a linear discrete-time basis function of the controller defined as

$$\beta = [\beta_1(z^{-1}), \beta_2(z^{-1}), \dots, \beta_{n_c}(z^{-1})]^T,$$

where  $n_c$  is the number of controller parameters.

By the Parseval's theorem, the time-domain cost function corresponding to  $J_{MR}(\theta)$  is described as

$$J_s(\theta) = \left\| W \left( \left[ M - \frac{PC(\theta)}{1+PC(\theta)} \right] r(t) - \frac{1}{1+PC(\theta)} v(t) \right) \right\|_2^2$$

where  $W$  is an appropriate filter which will be computed subsequently such that the criterion  $J_s(\theta)$  becomes a good approximation of  $J_{MR}(\theta)$ . If  $v(t) = 0$ ,  $J_s(\theta)$  is asymptotically equivalent to  $J_{MR}(\theta)$ . There are three difficulties in minimizing  $J_s(\theta)$  as follows:

- It is difficult to find the global minimum because  $J_s(\theta)$  is nonlinear for  $\theta$ .
- Multiple experiments are required to evaluate  $J_s(\theta)$  because the input data to the plant,  $u(\theta, t)$ , relies on  $\theta$ .
- $J_s(\theta)$  is not equivalent to  $J_{MR}(\theta)$  because  $v(t) \neq 0$  in many cases.

These problems are solved in the NCbT

### 2. Design Problem of NCbT

For the first difficulty, the convex approximation is introduced. Let  $C^*$  be an ideal controller which achieves a reference model  $M$ , that is

$$M = \frac{PC^*}{1+PC^*} \quad (3)$$

Assuming that a designed controller  $C(\theta)$  approaches to  $C^*$  enough,  $1/(1+PC(\theta)) \approx 1/(1+PC^*)$ .  $J_{MR}(\theta)$  is then approximated as  $J_c(\theta)$  by a convex function for  $\theta$  as

$$J_c(\theta) = \left\| M - (1-M)PC(\theta) \right\|_2^2 \quad (4)$$

For the second difficulty, the positions of  $P$  and  $C(\theta)$  are swapped to fix the input of  $P$  and not to depend on  $\theta$ . As a result, it is possible to evaluate  $J_c(\theta)$  with a single experimental data set.

For the third difficulty, the correlation approach is introduced. Define  $f(\theta)$  as

$$f(\theta) = \frac{1}{T} \sum_{t=1}^T [\zeta_w(t) \varepsilon_s(\theta, t)], \quad (5)$$

where  $T$  is the number of samples in one period of a reference signal  $r(t)$ , and  $\zeta_w(t)$  is a vector of instrumental variables correlated with  $r_w(t)$  and uncorrelated with  $v(t)$  given by

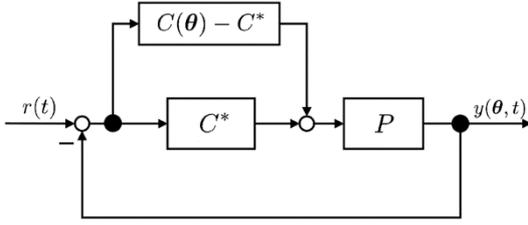


Fig. 3. Closed-loop system with an ideal controller

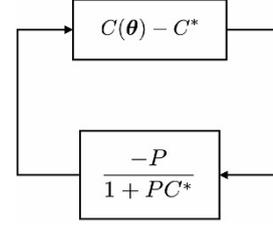


Fig. 4. Small gain theorem

$$\zeta_w(t) = [r_w(t+l), r_w(t+l-1), \dots, r_w(t-l)]^T$$

with  $r_w(t) = Wr(t)$  and  $l$  is a sufficient large integer. With an approximation of  $1 - M = 1/(1 + PC^*)$ , properly chosen  $W$  and sufficiently large  $l$ ,  $f^T(\theta)f(\theta)$  is asymptotically equivalent to  $J_{MR}(\theta)$ , and influence of noise  $v(t)$  uncorrelated with  $r(t)$  can be reduced. Since  $f^T(\theta)f(\theta)$  is convex for  $\theta$ , optimal parameters can be derived by solving the following convex optimization problem:

$$\begin{aligned} \hat{\theta} &= \arg \min_{\theta} J(\theta), \\ J(\theta) &= f^T(\theta)f(\theta). \end{aligned} \quad (6)$$

### 3. Stability Constraint Based on Small Gain Theorem

The controller  $C(\theta)$  designed by the NCbT does not always stabilize plants because  $C(\theta)$  is just designed to minimize  $J(\theta)$ . In this section, a stability constraint based on the small gain theorem is introduced to guarantee closed-loop stability in (Heusden et al., 2011). A closed-loop system consistin of a controller  $C(\theta)$  and a stable and minimum phase plant is represented by an ideal controller  $C^*$  and a controller perturbation  $C(\theta) - C^*$  as Fig. 3. Then, a stabilization problem of a closed-loop system by  $C(\theta)$  in Fig. 3 can be regarded as a robust stabilization problem for an additive perturbation  $C(\theta) - C^*$ . By equivalently converting Fig. 3 into Fig. 4 and assuming that  $C^*$  stabilizes  $P$ , the sufficient condition of stability of the closed-loop system based on the small gain theorem is given as

$$\delta(\theta) = \|G(\theta)\|_{\infty} < 1, \quad (7)$$

where

$$G(\theta) = \frac{-P(C(\theta) - C^*)}{1 + PC^*}$$

Define  $\varepsilon_s(\theta, t)$  as

$$\begin{aligned} \varepsilon_s(\theta, t) &= G(\theta)r(t) \\ &= Mr(t) - C(\theta)(1 - M)Pr(t) \\ &= Mr(t) - C(\theta)(1 - M)y_0(t), \end{aligned} \quad (8)$$

where  $G(\theta)$  can be regarded as a transfer function from  $r(t)$  to  $\varepsilon_s(\theta, t)$  in Fig. 2 and  $y_0(t)$  is the output data of the plant  $P$  when  $r(t)$  is applied. It is known that  $\delta(\theta)$  can be estimated by using spectral analysis with  $r(t)$  and  $\varepsilon_s(\theta, t)$  (Ljung, 1999) as

$$\begin{aligned} \hat{\delta}(\theta) &= \max_{\{\omega_n | \Phi_r(\omega_n) \neq 0\}} |\hat{G}(\theta)|, \\ \hat{G}(\theta) &= \frac{\hat{\Phi}_{r\varepsilon_s}(\theta, \omega_n)}{\hat{\Phi}_r(\omega_n)} \\ \omega_n &= 2\pi n / T, n = 0, 1, \dots, (T-1) / 2 \end{aligned} \quad (9)$$

where  $\hat{\Phi}_{r\varepsilon_s}(\theta, \omega_n)$  is a power cross-spectral density between  $r(t)$  and  $\varepsilon_s(\theta, t)$ ,  $\hat{\Phi}_r(\omega_n)$  is a power spectral density of  $r(t)$ .  $\hat{\Phi}_{r\varepsilon_s}(\theta, \omega_n)$  and  $\hat{\Phi}_r(\omega_n)$  can be calculated as

$$\hat{\Phi}_{r\varepsilon_s}(\omega_n) = \sum_{\tau=0}^{T-1} R_{r\varepsilon_s}(\theta, \tau) e^{-j\omega_n \tau}, \quad (10)$$

$$\hat{\Phi}_r(\omega_n) = \sum_{\tau=0}^{T-1} R_r(\theta, \tau) e^{-j\omega_n \tau}, \quad (11)$$

Where  $R_{r\varepsilon_s}(\theta, \tau)$  is a cross-correlation function between  $r(t)$  and  $\varepsilon_s(\theta, t)$  and  $R_r(t)$  is an auto-correlation function  $r(t)$  defined as

$$\hat{R}_{r\varepsilon_s}(\theta, \tau) = \frac{1}{T} \sum_{t=1}^T \varepsilon_s(\theta, t - \tau) r(t), \quad (12)$$

$$\hat{R}_r(\tau) = \frac{1}{T} \sum_{t=1}^T r(t - \tau) r(t), \quad (13)$$

As a result, the convex optimization problem of the NCbT to guarantee closed-loop stability is formulated as

$$\hat{\theta} = \arg \min_{\theta} J(\theta),$$

subject to

$$\hat{\delta}(\theta) < 1$$

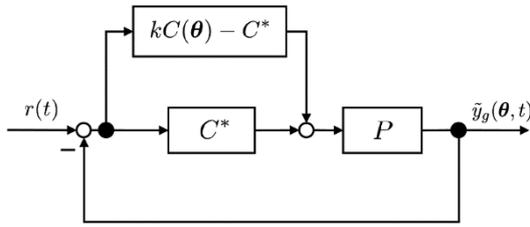


Fig. 5. Closed-loop system with gain fluctuation  $kP$

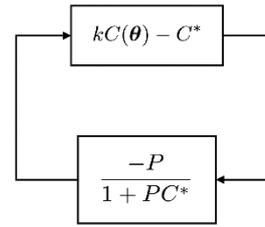


Fig. 6. Small gain theorem for gain fluctuation

The parameter derived by the above problem can stabilize the closed-loop system for the situation where input/output data set is collected, but does not always stabilize the closed-loop system when the plant is fluctuated. Therefore, the closed-loop system may be unstable by a little plant perturbation.

### III. PROPOSED METHOD

In model-based approaches, stability margins such as gain margin and phase margin are evaluated using the plant model. However, in data-driven approaches, it is difficult to evaluate stability margin since plants model is not available. It is also difficult to adjust controller parameter to establish a prespecified stability margin after optimization. In this section, constraints for data-driven approaches to guarantee gain and phase margins are proposed.

#### 1. Constraint for Gain Margin

If a controller stabilizes a gain-fluctuated plants  $kP$  ( $k > 0$ ) as well as  $P$ , gain margin of  $20\log k$  [dB] is at least established. To guarantee gain margin of  $20\log k$  [dB] in data-driven controller design methods,  $C(\theta)$  should stabilize not only  $P$  but also  $kP$ . The sufficient condition for stabilizing  $P$  is formulated in the previous section. In this subsection, we will formulate the constraint to stabilize  $kP$ , and then the convex optimization problem to guarantee gain margin.

To stabilize a perturbed plant  $kP$  by a controller  $C(\theta)$  is equivalent to stabilize  $P$  by a perturbed controller  $kC(\theta)$ . For a closed-loop system consisting of  $kC(\theta)$  and  $P$ , consider a controller perturbation from an ideal controller  $C^*$  as Fig. 5. Then, a stabilization problem of the closed-loop system in Fig. 5 can be regarded as a robust stabilization problem for the additive perturbation  $kC(\theta) - C^*$ . By equivalently converting Fig. 5 into Fig. 6 as similarly in the previous section, the sufficient condition of stability of the closed-loop system with  $kP$  based on the small gain theorem is given as

$$\delta_g(\theta) = \|G_g(\theta)\|_\infty < 1, \tag{14}$$

where

$$G_g(\theta) = \frac{-P(kC(\theta) - C^*)}{1 + PC^*}.$$

Define  $\varepsilon_g(\theta, t)$  as an output of  $G_g(\theta)$  when  $r(t)$  is applied:

$$\begin{aligned} \varepsilon_g(\theta, t) &= G_g(\theta)r(t) \\ &= Mr(t) - (1 - M)C(\theta)kPr(t) \\ &= Mr(t) - (1 - M)C(\theta)y_k(t). \end{aligned} \tag{15}$$

where  $y_k(t)$  is the output data of  $kP$  when  $r(t)$  is applied. If  $P$  is linear time-invariant and  $v(t) = 0$ ,  $y_k(t) = ky_0(t)$  for its linearity. We can compute  $\varepsilon_g(\theta, t)$  from  $r(t)$  and  $ky_0(t)$ , and then, we can estimate  $\delta_g(\theta)$  using spectral analysis with  $r(t)$  and  $\varepsilon_g(\theta, t)$  as

$$\begin{aligned} \hat{\delta}_g(\theta) &= \max_{\{\omega_n | \hat{\Phi}_r(\omega_n) \neq 0\}} |\hat{G}_g(\theta)|, \\ \hat{G}_g(\theta) &= \frac{\hat{\Phi}_{r\varepsilon_g}(\theta, \omega_n)}{\hat{\Phi}_r(\omega_n)} \\ \omega_n &= 2\pi n / T, \quad n = 0, 1, \dots, (T-1) / 2, \end{aligned} \tag{16}$$

where  $\hat{\Phi}_{r\varepsilon_g}(\theta, \omega_n)$  is an estimate of a power cross-spectral density between  $r(t)$  and  $\varepsilon_g(\theta, t)$ .  $\hat{\Phi}_{r\varepsilon_g}(\theta, \omega_n)$  can be calculated as

$$\hat{\Phi}_{r\varepsilon_g}(\theta, \omega_n) = \sum_{\tau=0}^{T-1} \hat{R}_{r\varepsilon_g}(\theta, \tau) e^{-j\omega_n \tau}, \tag{17}$$

where  $\hat{R}_{r\varepsilon_g}(\theta, \tau)$  is an estimate of a cross-correlation function between  $r(t)$  and  $\varepsilon_g(\theta, t)$  defined as

$$\hat{R}_{r\varepsilon_g}(\theta, \tau) = \frac{1}{T} \sum_{t=1}^T \varepsilon_g(\theta, t - \tau)r(t), \tag{18}$$

In many practical situations,  $v(t) \neq 0$ . However, since  $G_g(\theta)$  is estimated using a cross-correlation function between  $r(t)$  and  $\varepsilon_g(\theta, t)$ , the estimate of  $G_g(\theta)$  is enough accurate even when  $v(t) \neq 0$ . Note that  $k$  is an arbitrary positive constant to determine gain margin to be guaranteed. By determining  $k$ , the convex optimization problem of the NCbT to guarantee gain margin of  $20 \log k$  [dB] is formulated as

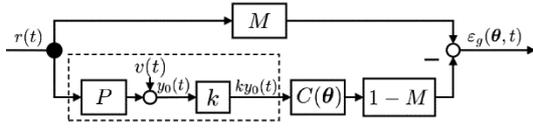


Fig. 7. Generation of  $\varepsilon_g(\theta, t)$  for gain fluctuation

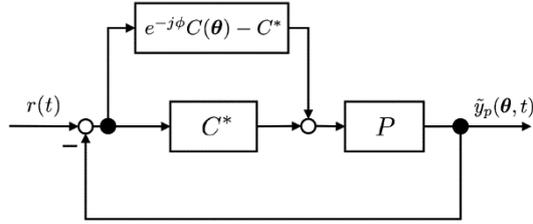


Fig. 8. Closed-loop system with phase lag  $e^{-j\phi}P$

$$\arg \min_{\theta} J(\theta),$$

Subject to

$$\hat{\delta}(\theta) < 1 \text{ and } \hat{\delta}_g(\theta) < 1.$$

## 2. Constraint for Phase Margin

A constraint for phase margin is also derived as same way in that for gain margin. If a controller stabilizes a phase-shifted plant  $e^{-j\phi}P$  as well as  $P$ , phase margin of  $\phi_d = \phi \times 180/\pi$  [deg] is at least established. To guarantee phase margin of  $\phi_d$  in data-driven controller design methods,  $C(\theta)$  should stabilize not only  $P$  but also  $e^{-j\phi}P$ , and then the convex optimization problem to guarantee phase margin. As similar to gain margin, to stabilize the phase-shifted plant  $e^{-j\phi}P$  by  $C(\theta)$  is equivalent to stabilize  $P$  by a phase-shifted controller  $e^{-j\phi}C(\theta)$ . For a closed-loop system consisting of  $e^{-j\phi}C(\theta)$  and  $P$ , consider a controller perturbation from an ideal controller  $C^*$  as Fig. 8. Then, a stabilization problem of a closed-loop system in Fig. 8 can be regarded as a robust stabilization problem for the additive perturbation  $e^{-j\phi}C(\theta) - C^*$ . We can notice that the constraint for phase margin is easily obtained by replacing  $kP$  and  $kC(\theta)$  in the previous subsection by  $e^{-j\phi}P$  and  $e^{-j\phi}C(\theta)$ , respectively. By equivalently converting Fig. 8 into Fig. 9, the sufficient condition for stability of the closed-loop system with  $e^{-j\phi}P$  based on the small gain theorem is given as

$$\delta_p(\theta) = \|G_p(\theta)\|_{\infty} < 1 \quad (19)$$

where

$$G_p(\theta) = \frac{-P(e^{-j\phi}C(\theta) - C^*)}{1 + PC^*}.$$

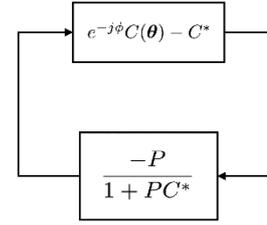


Fig. 9. Small gain theorem for phase lag

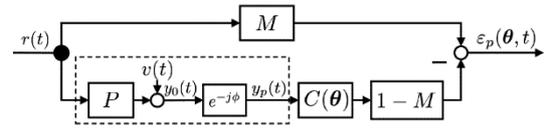


Fig. 10. Generation of  $\varepsilon_p(\theta, t)$  for phase fluctuation

Define  $\varepsilon_p(\theta, t)$  as an output of  $G_p(\theta)$  when  $r(t)$  is applied:

$$\begin{aligned} \varepsilon_p(\theta, t) &= G_p(\theta)r(t) \\ &= Mr(t) - (1-M)C(\theta)e^{-j\phi}Pr(t) \\ &= Mr(t) - (1-M)C(\theta)y_p(t) \end{aligned} \quad (20)$$

where  $y_p(t)$  is the output data of the phase-shifted plant  $e^{-j\phi}P$  when  $r(t)$  is applied. The problem is how to estimate  $y_p(t)$  using  $y_0(t)$ . Since  $y_p(t)$  is the output of  $e^{-j\phi}P$ , we should manipulate  $y_0(t)$  not in time-domain but in frequency-domain, that is, its Fourier transform  $Y_0(\omega)$ , where

$$\begin{aligned} Y_0(\omega) &= \sum_{t=0}^{T-1} y_0(t)e^{-j\omega_n t}, \\ \omega_n &= 2\pi n/T, \quad n = 0, 1, \dots, (T-1)/2. \end{aligned}$$

$y_p(t)$  is then calculated as the inverse Fourier transform of  $e^{-j\phi}Y_0(\omega)$ . Finally, we can estimate  $\delta_p(\theta)$  using spectral analysis with  $r(t)$  and  $\varepsilon_p(\theta, t)$  as

$$\begin{aligned} \hat{\delta}_p(\theta) &= \max_{\{\omega_n | \hat{\Phi}_r(\omega_n) \neq 0\}} |\hat{G}_p(\theta)|, \\ \hat{G}_p(\theta) &= \frac{\hat{\Phi}_{r\varepsilon_p}(\theta, \omega_n)}{\hat{\Phi}_r(\omega_n)}, \\ \omega_n &= 2\pi n/T, \quad n = 1, 2, \dots, (T-1)/2, \end{aligned} \quad (21)$$

where  $\hat{\Phi}_{r\varepsilon_p}(\theta, \omega_n)$  is an estimate of a power cross-spectral density between  $r(t)$  and  $\varepsilon_p(\theta, t)$ .  $\hat{\Phi}_{r\varepsilon_p}(\theta, \omega_n)$  can be calculated as

$$\hat{\Phi}_{r\varepsilon_p}(\theta, \omega_n) = \sum_{\tau=0}^{T-1} \hat{R}_{r\varepsilon_p}(\theta, \tau) e^{-j\omega_n \tau}, \quad (22)$$

where  $\hat{R}_{r\varepsilon_p}(\boldsymbol{\theta}, \tau)$  is an estimate of a cross-correlation function between  $r(t)$  and  $\varepsilon_p(\boldsymbol{\theta}, t)$  defined as

$$\hat{R}_{r\varepsilon_p}(\boldsymbol{\theta}, \tau) = \frac{1}{T} \sum_{t=1}^T \varepsilon_p(\boldsymbol{\theta}, t - \tau) r(t), \quad (23)$$

As stated in estimating  $y_k(t)$ ,  $v(t)$  does not affect the estimate of  $y_p(t)$  due to the same reason in estimating  $y_k(t)$ . Note that  $\phi$  is an arbitrary constant to determine phase margin to be guaranteed. By determining  $\phi$ , the convex optimization problem of the NCbT to guarantee phase margin of  $\phi_d$  [deg] is formulated as

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$$

subject to

$$\hat{\delta}(\boldsymbol{\theta}) < 1 \text{ and } \hat{\delta}_p(\boldsymbol{\theta}) < 1$$

### 3. Constraints to Guarantee Gain and Phase Margins

Gain and phase margins are often simultaneously imposed in actual design problems. Note that the above mentioned constraints for gain and phase margins are convex with respect to the controller parameter  $\boldsymbol{\theta}$ . The convex optimization problem of the NCbT to simultaneously guarantee gain margin of  $20 \log k$  [dB] and phase margin of  $\phi_d$  [deg] is formulated as

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}),$$

$$\hat{\delta}(\boldsymbol{\theta}) < 1, \hat{\delta}_g(\boldsymbol{\theta}) < 1, \text{ and } \hat{\delta}_p(\boldsymbol{\theta}) < 1.$$

### 4. Robustness Issue

In practical applications, simultaneous fluctuations of gain and phase are often encountered. Since gain/phase margin just guarantees closed-loop stability only for single fluctuation of gain/phase, the closed-loop system is not always stabilized for simultaneous fluctuations of gain and phase even for small changes of gain and phase. In this subsection, a stabilization problem for simultaneous fluctuations of gain and phase is addressed for more robustness by extending the constraints for gain and phase margins described in the previous subsections.

Consider simultaneously fluctuated plant,  $ke^{-j\phi}P$  ( $1 \leq k \leq kr$ ,  $0 \leq \phi \leq \phi_r$ ). In order to stabilize  $ke^{-j\phi}P$  for all  $k$  and  $\phi$ , an infinite number of constrains are required, which is not intractable. A practical solution for this difficulty is to divide 2 closed intervals,  $[1, kr]$  and  $[0, \phi_r]$ , for  $k$  and  $\phi$  into enough small regions and to impose gain and phase constraints for all combination of a pair  $(k, \phi)$ . Define a gridded point  $k_{n_g}$  and  $\phi_{n_p}$  as

$$k_{n_g} = \frac{(k_r - 1)n_g + N_g}{N_g}, \quad n_g = 0, 1, \dots, N_g \text{ and}$$

$$\phi_{n_p} = \frac{n_p}{N_p} \phi_r, \quad n_p = 0, 1, \dots, N_p,$$

where enough large integers,  $N_g$  and  $N_p$ , denote the number of gridded points for the intervals for  $k$  and  $\phi$ , respectively. Then a stabilization problem for simultaneous fluctuations of gain and phase is approximated by that for  $k_{n_g} e^{-j\phi_{n_p}} P$  for all  $n_g$  and  $n_p$ . As similar way in constraints for gain and phase margins, the sufficient condition of stability of the closed-loop system with  $k_{n_g} e^{-j\phi_{n_p}} P$  based on the small gain theorem is given as

$$\delta_{n_g, n_p}(\boldsymbol{\theta}) = \|G_{n_g, n_p}(\boldsymbol{\theta})\|_{\infty} < 1, \quad (24)$$

where

$$G_{n_g, n_p}(\boldsymbol{\theta}) = \frac{-P(k_{n_g} e^{-j\phi_{n_p}} C(\boldsymbol{\theta}) - C^*)}{1 + PC^*},$$

$$n_g = 0, 1, \dots, N_g, n_p = 0, 1, \dots, N_p.$$

Note that Eq.(24) is a straightforward extension of the constraints in the former subsections, and includes Eqs.(14) and (19) as special cases. Define  $\varepsilon_{n_g, n_p}(\boldsymbol{\theta}, t)$  as an output of  $G_{n_g, n_p}(\boldsymbol{\theta})$  when  $r(t)$  is applied:

$$\begin{aligned} \varepsilon_{n_g, n_p}(\boldsymbol{\theta}, t) &= G_{n_g, n_p}(\boldsymbol{\theta}) r(t) \\ &= Mr(t) - (1 - M)C(\boldsymbol{\theta}) k_{n_g} e^{-j\phi_{n_p}} Pr(t) \\ &= Mr(t) - (1 - M)C(\boldsymbol{\theta}) y_{n_g, n_p}(t), \end{aligned} \quad (25)$$

where  $y_{n_g, n_p}(t)$  is the output data of  $k_{n_g} e^{-j\phi_{n_p}} P$ . The estimate of  $y_{n_g, n_p}(t)$  is obtained as the inverse Fourier transform of  $k_{n_g} e^{-j\phi_{n_p}} Y_0(\omega)$ .  $\hat{\delta}_{n_g, n_p}$  can be estimated as similar in constraints for gain and phase. Finally, the convex optimization problem of the NCbT to guarantee simultaneous fluctuations of gain and phase is formulated as

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}),$$

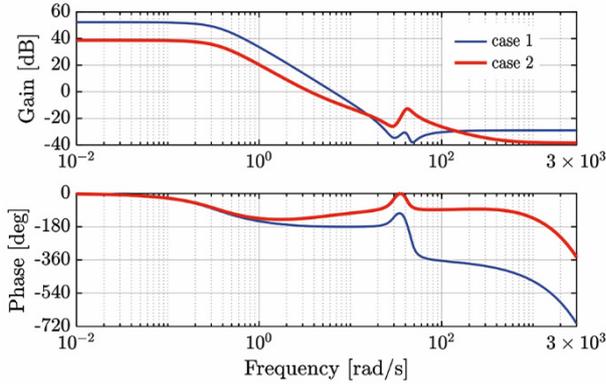
subject to

$$\begin{aligned} \hat{\delta}_{n_g, n_p}(\boldsymbol{\theta}) &< 1, \\ n_g &= 0, 1, \dots, N_g, n_p = 0, 1, \dots, N_p. \end{aligned}$$

Note that  $\hat{\delta}_{n_g, n_p}$  is, of course, convex with respect to  $\boldsymbol{\theta}$ .

**Table 1. Design Results for Simulation I**

	case 1	case 2
$J(\theta)$	0.4907	0.4930
$\hat{\delta}(\theta)$	0.9999	0.9956
$\hat{\delta}_g(\theta)$	1.0019	0.9999
$\hat{\delta}_p(\theta)$	1.0193	0.9999
GM [dB]	0.728	38.4
PM [deg]	0.0764	48.2

**Fig. 11. Bode plots of  $L(\theta)$  for Simulation I**

#### IV. NUMERICAL SIMULATIONS

In this section, we show the effectiveness of the proposed method through 2 numerical simulations: The first one is comparison of the design results with and without constraints for gain and phase margins. The second one shows the design result for simultaneous fluctuations of gain and phase of the plant to establish enough robustness.

##### 1. Simulation I

The reference model  $M$  and the plant  $P$  are given as

$$M = \frac{0.6321}{z - 0.3679},$$

$$P = \frac{0.069343(z^2 - 1.989z + 0.9901)A_1(z)}{(z^2 - 1.989z + 0.9902)B_1(z)},$$

$$A_1(z) = (z - 0.9953)(z - 0.9953), \text{ and}$$

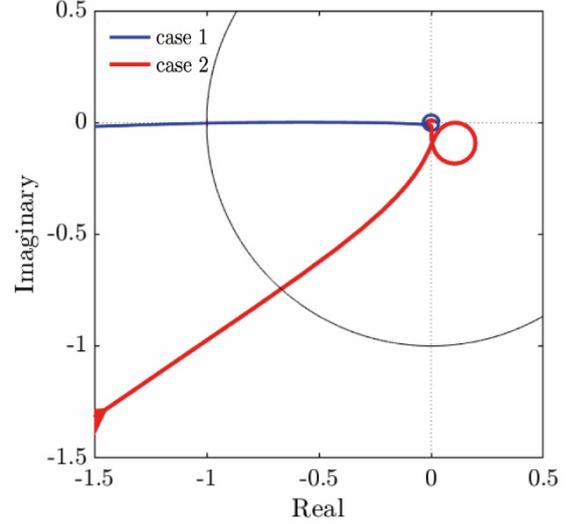
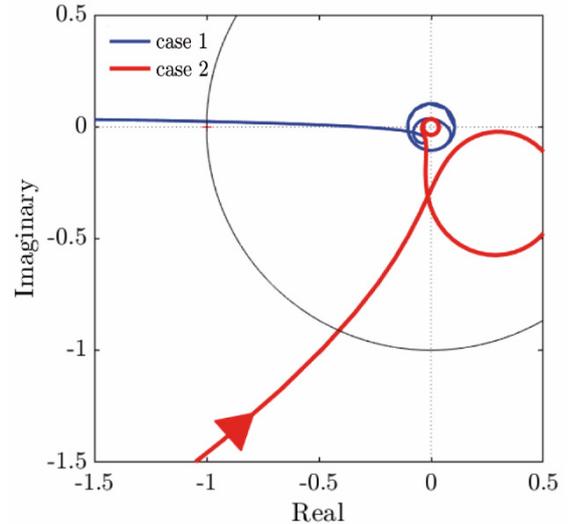
$$B_1(z) = z(z - 0.995)(z - 0.9971)(z - 0.9993).$$

Note that  $P$  is only used to acquire the input/output data set for design. The linearly parameterized controller  $C(\theta)$  is defined as

$$C(\theta) = \beta^T \theta,$$

$$\theta = [\theta_p \quad \theta_i \quad \theta_d]^T, \text{ and}$$

$$\beta = \begin{bmatrix} 1 & \frac{T_s}{1-z^{-1}} & \frac{1-z^{-1}}{T_s} \end{bmatrix}^T,$$

**Fig. 12. Vector locus of  $L(\theta)$  for Simulation****Fig. 13. Vector locus of  $L(\theta)$  for Simulation I**

which is a discrete-time PID controller. The sampling time  $T_s$  is set to 1 ms. Control requirements for gain and phase-margins,  $k$  and  $\phi$ , are set as  $k = \sqrt{10}$  and  $\phi = 2\pi/9$  which are determined to establish 10 dB of gain margin and 40 deg of phase margin, respectively. For data acquisition, 3 periods of a pseudo random binary signal (PRBS) of length  $N = 3 \times (2^{14} - 1) = 49149$  is applied to the plant.

We refer to simulations with and without constraints for gain and phase margins as case 1 and case 2, respectively. Only stability constraint  $\delta(\theta) < 1$  is imposed in case 1, while  $\delta_g(\theta) < 1$  and  $\delta_p(\theta) < 1$  are imposed as well as  $\delta(\theta) < 1$  in case 2. The tuning results of case 1 and 2 are shown in Table 1. The Bode plots of the open-loop transfer function  $L(\theta) = PC(\theta)$  are shown in Fig. 11. Table 1 shows that the evaluated value  $J(\theta)$  of case 2 is larger than that of case 1. In Fig. 11, we can

**Table 2. Design Results for Simulation II**

	case 1	case 2	case 3
$J(\theta)$	$7.2 \times 10^{-4}$	$7.5 \times 10^{-4}$	$1.7 \times 10^{-3}$
GM [dB]	47.0	46.0	43.4
PM [deg]	49.0	50.9	59.6

see that the gain-crossover frequency of case 2 is lower than that of case 1. However, case 1 does not achieve  $\delta_g(\theta) < 1$  and  $\delta_p(\theta) < 1$ , while case 2 does. Fig. 12 shows that the vector locus of  $L(\theta)$  of case 1 has a little stability margin, while that of case 2 has enough gain and phase margins. From Table 1. and Fig. 12, we can confirm that the proposed constraints guarantee 10 dB of gain margin and 40 deg of phase margin, which means that the specifications for stability margins are satisfied by the proposed data-driven constraints.

The fluctuated plant is given as  $\tilde{P} = 3\mathcal{Z}^{-2} \times P$  to check robustness for the plant fluctuation. The vector loci of the fluctuated open-loop transfer function  $\tilde{L}(\theta) = \tilde{P}C(\theta)$  are shown in Fig. 13. Fig. 13 shows that the vector locus of  $\tilde{L}(\theta)$  for case 1 is over the critical point while that of case 2 is not. The controller of case 1 destabilizes the closed-loop system with the fluctuated plant, but that of case 2 does not. We can confirm that the proposed method improves robustness for the plant fluctuation.

## 2. Simulation II

The reference model  $M$  and the plant  $P$  are given as

$$M = \frac{0.0080079(z-0.99)}{(z^2-1.992z+0.992)},$$

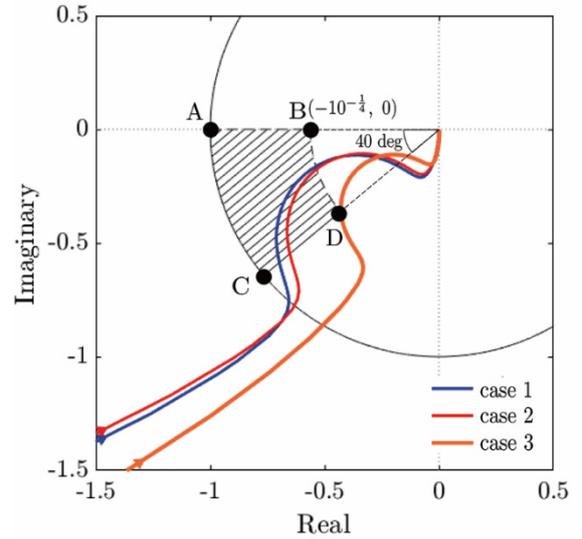
$$P = \frac{0.069347A_2(z)(z^2-1.995z+0.9956)}{B_2(z)(z^2-1.995z+0.9955)},$$

$$A_2(z) = (z-0.996)(z-0.9953), \text{ and}$$

$$B_2(z) = (z-0.9971)(z-0.995)(z-0.9993).$$

$k_r$  and  $\phi_r$  are set to  $k_r = 10^{1/4}$  and  $\phi_r = 2\pi/9$  to establish gain margin of 5 dB and phase margin of 40 deg.  $N_g = N_p = 8$  and data length of the reference signal  $r(t)$  is given as  $N = 4 \times (2^{14} - 1) = 65532$ , which corresponds to 4 periods of PRBS. The other design conditions are same as in Simulation I. In Simulation II, case 3 is evaluated for the case where  $\delta_{n_g, n_p}(\theta) < 1$  for all combination of  $(n_g, n_p)$  in addition to case 1 and 2.

The tuning results for all cases are listed in Table 2. Requirements for gain and phase margins are established for all cases. Fig. 14 shows the vector loci of  $L(\theta)$  for all cases. In Fig. 14, the coordinates of the points A, B, C, and D are  $(-1, 0)$ ,  $(-10^{-1/4}, 0)$ ,  $(-\cos(2\pi/9), -\sin(2\pi/9))$ , and  $(-10^{-1/4}\cos(2\pi/9), -10^{-1/4}\sin(2\pi/9))$ , respectively. The vector loci of  $L(\theta)$  for case 1 and 2 intersect the shaded area surrounded by 4 points on the complex plane, A, B, C, and D. This

**Fig. 14. Vector locus of  $L(\theta)$  for Simulation II**

means that the control system might be destabilized even when the simultaneous changes in gain and phase smaller than 5 dB and 40 deg, respectively occurs. The simultaneous changes of 5 dB in gain and of 40 deg in phase clearly destabilize the closed-loop system for case 1 and 2. In case 1, since the stability constraint is only imposed, the vector locus of  $L(\theta)$  is shaped so as to avoid the point A (critical point). The vector locus of  $L(\theta)$  for case 2 is also shaped so as to avoid the points B and C as well as the point A. On the other hand, the vector locus of  $L(\theta)$  for case 3 does not enter the shaded area, which means that the control system establishes enough stability margins for simultaneous gain and phase changes. In fact, the vector locus of  $L(\theta)$  does not intersect the point A even when the simultaneous changes of 5 dB in gain and of 40 deg in phase occurs. This is because the constraints are imposed for all combination of  $(n_g, n_p)$  finely gridded in the shaded area. As a result, the vector locus of  $L(\theta)$  is shaped so as to avoid the shaded area. Case 3 achieves higher robustness at the expense of  $J(\theta)$ .

## V. CONCLUSIONS

This paper proposed the constraints to guarantee arbitrary gain and phase margins for data-driven controller tuning methods. These constraints are extended easily from the standard stability constraint based on the small gain theorem. The effectiveness of the proposed method is verified through the simulation examples. As future works, implementation on the real applications and reduction of conservativeness caused by the small gain theorem should be tackled.

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