NATURAL BOUNDARY ELEMENT METHOD FOR ELLIPTIC BOUNDARY VALUE PROBLEMS IN DOMAINS WITH CONCAVE ANGLE

Qi-Kui Du* and De-Hao Yu**

Keywords: natural boundary reduction (NBR), concave angle domain, numerical implementation, coupling.

ABSTRACT

In this paper, the natural boundary reduction for some elliptic boundary value problems with concave angle domains and its natural boundary element methods are investigated. Natural integral equations and Poisson integral formulae are given. A finite element methods of natural integral equations are discussed in details. The convergence of approximate solutions and their error estimates are obtained. Some numerical experiments are presented to demonstrate the performance of the method and our estimates. As an application, we present the coupling of FEM and natural boundary element.

INTRODUCTION

In this paper, we consider a kind of elliptic problems with concave angle domains in two dimensions, which is Neumann boundary-value problem. Let \( \Omega \) and \( \Omega' \) are a bounded sector domain with angle \( \alpha \) and an exterior concave angle domains, respectively, and \( 0 < \alpha \leq 2\pi \).

We consider some linear elliptic second-order boundary value problems in two dimensions. The boundary of domain \( \Omega \) or \( \Omega' \) is decomposed into three disjoint parts, \( \Gamma \), \( \Gamma_0 \) and \( \Gamma_\alpha \) where a Neumann boundary conditions are given. The statement of the problems considered is:

\[
\Delta u + \beta u = 0, \quad \text{in } \Omega \text{ or } \Omega' \quad (1)
\]

\[
\frac{\partial u}{\partial n}(r, 0) = 0, \quad \text{on } \Gamma_0 \quad (2)
\]

\[
\frac{\partial u}{\partial n}(r, \alpha) = 0, \quad \text{on } \Gamma_\alpha
\]

\[
\frac{\partial u}{\partial n}(R, \theta) = g_n(R, \theta), \quad \text{on } \Gamma
\]

If domain is \( \Omega' \), some conditions at infinity. \( u \) is the unknown function, \( \frac{\partial u}{\partial n} \) is the normal derivative of \( u \) on boundary, \( g_n(R, \theta) \) is given function, and \( \beta \) is a constant. Equation (1) is Laplace, Helmholtz or modified Helmholtz equation, according as \( \beta \) is zero, positive or negative. In this paper we shall concentrate on the method of natural boundary element method [1, 2, 3, 4] or DtN method [5, 6]. It is well-known that the natural integral equation is hypersingular, its kernel function has non-integrable singularity, the integral is to be understood in the sense of the theory of distribution. It can be referred to [10]. The condition at infinity (5) is as follows. If \( \beta \leq 0 \) then the solution \( u \) is required to vanish at infinity; If \( \beta > 0 \) then at infinity the solution \( u \) is required to be imposed a radication condition

\[
\lim_{r \to +\infty} r^{\beta/2} \frac{\partial u}{\partial r} = 0
\]

In the above problems, Yu [1] has only investigated the problems for \( \beta = 0 \). However, there are some more significant engineering background for \( \beta \neq 0 \), such as waveguide, electromagnetic radiation, geophysics, meteorology, etc. In addition, discretization in time for some time-dependent problems is reduced to the model (1) [7]. So, it has been of great importance in theory and practical applications to investigate some numerical methods of the above problems systematically.

Givoli, Rivkin and Keller [11], Wu and Han [12] used a method called the DtN finite-element method to some elliptic boundary value problems in domains with corners and singularities. They gave a sequence of approximations to the exact boundary conditions on an artificial boundary by using Green’s function. On the boundary, the Neumann boundary condition is related to the Dirichlet boundary condition, or the second-order
derivative of the solution on the artificial boundary. Bao and Han [13] gave a high-order local approximate artificial boundary condition. Accurate numerical results were obtained for this approach in [11, 12]. In this paper, we follow the idea of [2, 5, 10] for solving some elliptic boundary value problems in domains (bounded and unbounded) with concave angle, and give their natural boundary reduction and natural boundary element method (or called the DtN finite-element method).

The outline of the paper is as follows. In Section 2 we derive natural integral equations on Ω and Poisson integral equations in Ω or Ω'. We consider the Laplace, Helmholtz and modified Helmholtz equations in Ω or Ω'. In Section 3 we discuss the numerical implementations of natural integral equations by finite element method. Section 4 contains some error analyses for the finite element scheme. To demonstrate the performance of the method, we present in Section 5 some numerical experiments and examples. As an application, we also discuss the coupling of FEM and NBE in infinite sector domain in Section 6. We close with concluding remarks in Section 7.

NATURAL BOUNDARY REDUCTION

For problems (1)-(5), the function \( g_{\alpha}(R, \theta) \) satisfies compatibility condition \( \int_{0}^{\pi} g_{\alpha}(R, \theta) d\theta = 0 \) if \( \beta = 0 \).

1. Bounded Sector Domain

Domain \( \Omega \), boundaries \( \Gamma \), \( \Gamma_0 \) and \( \Gamma_\alpha \) are as follows: \( \Omega := (r, \theta) \mid 0 < r < R, \theta \in (0, \alpha) \), \( \Gamma := \{(r, \theta) \mid \theta \in (0, \alpha)\} \), \( \Gamma_0 := \{(r, 0) \mid 0 \leq r \leq R\} \), \( \Gamma_\alpha := \{(r, \alpha) \mid 0 \leq r \leq R\} \).

By separation of variables we find the general solution which satisfies (1) in the domain \( \Omega \) and the conditions mentioned in section 1:

\[
 u(r, \theta) = \sum_{n=0}^{\infty} A_n \cdot B_n(r) \cos \frac{n \pi \theta}{\alpha}, \quad r < R \quad (7)
\]

The coefficients \( B_n(r) \) for all different cases of \( \beta \) considered are summarized in Table 1. There \( I_n(x) \) is the modified Bessel function of the first kind of order \( v \), and \( J_n(x) \) is the Bessel function of order \( v \) \([8, 9]\), i.e.,

\[
 I_n(x) = \sum_{j=0}^{\infty} \frac{1}{j! \cdot (j + 1 + v)} \left( \frac{x}{2} \right)^{2j + v},
\]

\[
 J_n(x) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j! \cdot (j + 1 + v)} \left( \frac{x}{2} \right)^{2j + v}.
\]

Table 1. The coefficients \( B_n(r) \) for all different cases of \( \beta \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \beta = 0 )</th>
<th>( \beta &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_n(r) )</td>
<td>( I_n \sqrt{\beta} r )</td>
<td>( r^{\frac{n}{\alpha}} )</td>
</tr>
</tbody>
</table>

From the following

\[
 u(R, \theta) = \lim_{r \to R} u(r, \theta) = \sum_{n=0}^{\infty} A_n B_n(R) \cos \frac{n \pi \theta}{\alpha} \quad (8)
\]

Using the orthogonality of the \( \text{cosines} \), we obtain the coefficients \( A_n, n = 0, 1, 2, \ldots \),

\[
 A_n = \frac{1}{\alpha} \epsilon_n B_n(R)^{-1} \int_{0}^{\alpha} u(R, \theta) \cos \frac{n \pi \theta}{\alpha} d\theta \quad (9)
\]

Hence

\[
 u(r, \theta) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \epsilon_n B_n(r) \cdot B_n(R)^{-1} \int_{0}^{\alpha} u(R, \theta') \cos \frac{n \pi \theta}{\alpha} \cdot \cos \frac{n \pi \theta'}{\alpha} d\theta', \quad r < R \quad (10)
\]

Where \( R \) is the radius of circular arc \( \Gamma \). If we let

\[
 G_n(r, \theta) := B_n(r) \cdot R_n(R)^{-1}
\]

then (10) can be expressed as follows:

\[
 u(r, \theta) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \epsilon_n G_n(r; R) \int_{0}^{\alpha} u(R, \theta) \cos \frac{n \pi \theta}{\alpha} \cdot \cos \frac{n \pi \theta'}{\alpha} d\theta', \quad r < R \quad (11)
\]

We differentiate (11) with respect to \( r \) and take the limit as \( r \) approaches to \( R' \), to obtain

\[
 \frac{\partial u}{\partial r} = \frac{\pi \alpha}{R \alpha^2} \sum_{n=0}^{\infty} \epsilon_n Z_n \int_{0}^{\alpha} u(R, \theta) \cos \frac{n \pi \theta}{\alpha} \cdot \cos \frac{n \pi \theta'}{\alpha} d\theta' \quad (12)
\]

The coefficients \( G_n(r, R) \) and \( Z_n \) in (11) and (12) are summarized in Table 2. And \( \epsilon_n \) is: \( \epsilon_n = 1 \), if \( n = 0 \). \( \epsilon_n = 2 \), if \( n \neq 0 \).

Table 2. The coefficients \( G_n(r, R) \) and \( Z_n \) for all different cases of \( \beta \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( G_n(r, R) )</th>
<th>( Z_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta &lt; 0 )</td>
<td>( I_n \sqrt{\beta} r )</td>
<td>( \alpha \sqrt{\beta} ) ( \frac{1}{\pi R} \cdot \frac{I_n}{I_n} \sqrt{\beta} R )</td>
</tr>
<tr>
<td>( \beta = 0 )</td>
<td>( \frac{r^{n \pi}}{\pi} )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \beta &gt; 0 )</td>
<td>( J_n \sqrt{\beta} r )</td>
<td>( \alpha \sqrt{\beta} ) ( \frac{1}{\pi R} \cdot \frac{J_n}{J_n} \sqrt{\beta} R )</td>
</tr>
</tbody>
</table>
2. Infinite Sector Domain

Domain $\Omega^*$, boundaries $\Gamma$, $\Gamma_0$ and $\Gamma_{\omega}$ are as follows:

$$\Omega^* := \{(r, \theta) \mid r > R, \theta \in (0, \omega)\}, \quad \Gamma := \{(R, \theta) \mid \theta \in (0, \omega)\}, \quad \Gamma_0 := \{(r, 0) \mid r > R\}, \quad \Gamma_{\omega} := \{(r, \omega) \mid r > R\}.$$

Similar to the above methods, we get the solution of the problem (1)-(5) for all cases of $\beta$ considered

$$u(r, \theta) = \frac{1}{\Gamma} \int_0^\infty e_n G_n(r, R) \int_0^\infty u(R, \theta) \cos \frac{n \pi \theta}{\alpha} \cos \frac{n \pi \theta}{\alpha} d\theta$$

$$\equiv \Delta u(R, \theta), \quad r > R$$

(13)

Furthermore

$$\frac{\partial u}{\partial r} = -\frac{\pi}{R \alpha^2} \int_0^\infty e_n Z_n \int_0^\infty u(R, \theta) \cos \frac{n \pi \theta}{\alpha} \cos \frac{n \pi \theta}{\alpha} d\theta,$$ on $\Gamma$

(14)

The coefficients $G_n(r, R)$ and $Z_n$ in (13) and (14) are summarized in Table 3. There $K_r(x)$ is the modified Bessel function of the second kind of order $v$, and $H^{(1)}_v(x)$ is the Hankel function of the first kind of order $v$ [8, 9], i.e.,

$$K_r(x) = \frac{\pi}{2} \frac{I_1(x) - I_n(x)}{\sin \pi v},$$

$$H^{(1)}_v(x) = \frac{i}{\sin \pi v} [J_n(x) \cdot e^{-i \pi v} - J_n(x)]$$

Since we have known that $\frac{d}{dn} = \frac{\partial}{\partial r}$ for interior problem, and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ for exterior problem, on $\Gamma$.

Therefore (12) and (14) can be written as follows:

$$\frac{\partial u}{\partial n} = \frac{\pi}{R \alpha^2} \int_0^\infty e_n Z_n \int_0^\infty u(R, \theta) \cos \frac{n \pi \theta}{\alpha} \cos \frac{n \pi \theta}{\alpha} d\theta$$

$$\equiv \Delta u(R, \theta), \quad \text{on } \Gamma$$

(15)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$G_n(r, R)$</th>
<th>$Z_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;0$</td>
<td>$\frac{K_\alpha}{K_\alpha^{\alpha/\sqrt{\beta}}}$</td>
<td>$\frac{K_\alpha^{\alpha/\sqrt{\beta}}}{\alpha} \cdot \frac{\alpha}{\sqrt{\beta}}$</td>
</tr>
<tr>
<td>$=0$</td>
<td>$\frac{R}{\alpha}$</td>
<td>$\frac{\alpha}{\sqrt{\beta}}$</td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$\frac{H_\alpha^{\alpha/\sqrt{\beta}}}{\sqrt{\beta}}$</td>
<td>$\frac{\alpha}{\sqrt{\beta}}$</td>
</tr>
</tbody>
</table>

Table 3. The coefficients $G_n(r, R)$ and $Z_n$ for all different cases of $\beta$

The coefficients $Z_n$ in (15) are given by Tables 2 and 3, respectively. That is, the coefficients $Z_n$ are given in Table 2 for the problem in bounded sector domain, and the coefficients $Z_n$ are given in Table 3 for the problem in infinite sector domain. Equations (11) and (13) are often called the Poisson integral formulae, and equation (15) is called natural integral equation [10] or Dirichlet-to-Neumann(DtN) map [8, 9]. In practice, the natural integral equation is truncated after a finite number of terms, $M$, namely,

$$\frac{\partial u}{\partial n} = \frac{\pi}{R \alpha^2} \int_0^\infty e_n Z_n \int_0^\infty u(R, \theta) \cos \frac{n \pi \theta}{\alpha} \cos \frac{n \pi \theta}{\alpha} d\theta,$$ on $\Gamma$

(16)

To cope with the numerical analysis, referring to [10] we recall an equivalent definition of Sobolev space $H^s(\Gamma)$ for any real number $s$:

$$\forall f \in H^s(\Gamma) \Rightarrow f(R, \theta) = \sum_{n=-\infty}^{\infty} e_n f_n \cdot (e^{i \pi \theta} + e^{-i \pi \theta})$$

and

$$\sum_{n=-\infty}^{\infty} \left| 1 + \left( \frac{n \pi}{\alpha} \right)^2 \right| \cdot \left| e_n f_n \right|^2 < +\infty$$

where $\Gamma : = \{(R, \theta) : 0 < \theta \leq 2 \pi, \ R > 0\}$, and $f_n = \frac{1}{2\alpha} \int_\Gamma f(R, \theta) \cdot (e^{i \pi \theta} + e^{-i \pi \theta}) d\theta$ . Thus we assign the following norm on $H^s(\Gamma)$:

$$\left| f(R, \theta) \right|^2_{L^1(\Gamma)} := \sum_{n=-\infty}^{\infty} \left| 1 + \left( \frac{n \pi}{\alpha} \right)^2 \right| \cdot \left| e_n f_n \right|^2$$

Especially, if $s = 0$ we have

$$\left| f(R, \theta) \right|^2_{L^1(\Gamma)} := \sum_{n=-\infty}^{\infty} \left| e_n f_n \right|^2 = \left| f(R, \theta) \right|^2_{L^2(\Gamma)}.$$
Then the weak form of (15) is:

\[ (P) \text{ Find } u_0 \in H^\frac{1}{2}(\Gamma) \text{ such that } \]

\[ \tilde{D}(u_0, v_0) = \tilde{F}_n(v_0), \quad \forall v_0 \in H^\frac{1}{2}(\Gamma) \]  

(19)

**Lemma 1.** If \( u = \sum_{n=-\infty}^{\infty} u_n e^{int\theta} \), \( v = \sum_{n=-\infty}^{\infty} v_n e^{int\theta} \), then

\[ u \ast v = \int_0^\alpha u(R, \theta - \theta') \cdot \int_0^\alpha (R, \theta') d\theta' \]

\[ = \sum_{n=-\infty}^{\infty} (2\alpha u_n \cdot v_n) (e^{in\theta} + e^{-in\theta}) \]

**Proof** It is not difficult to obtain that the following

\[ \int_0^\alpha (e^{-in\theta} + e^{in\theta}) \cdot (e^{in\theta} + e^{-in\theta}) \theta d\theta \]

\[ = \begin{cases} 2\alpha, & m = \pm n \\ 0, & \text{otherwise} \end{cases} \]

Thus

\[ u \ast v = \int_0^\alpha u(R, \theta - \theta') \cdot v(R, \theta') d\theta' \]

\[ = \sum_{n=-\infty}^{\infty} u_n e^{in\theta} \sum_{n=-\infty}^{\infty} v_n e^{-in\theta} \int_0^\alpha (e^{in\theta} + e^{-in\theta}) \theta d\theta \]

\[ = \sum_{n=-\infty}^{\infty} (2\alpha u_n \cdot v_n) (e^{in\theta} + e^{-in\theta}) \]

This completes the proof.

From Lemma 1, we have

\[ \mathcal{K}u(R, \theta) = \frac{\pi}{R\alpha} \sum_{n=-\infty}^{\infty} e_n Z_n u_n \cdot (e^{in\theta} + e^{-in\theta}) \quad (20) \]

where \( u_n = \frac{1}{2\alpha} \int_0^\alpha u(R, \theta) \cdot (e^{in\theta} + e^{-in\theta}) d\theta \).

**Theorem 1.** For all non-negative real number \( s \), natural integral operator \( \mathcal{K} \) is a continuous linear operator from \( H^{s+\frac{1}{2}}(\Gamma) \) to \( H^{s-\frac{1}{2}}(\Gamma) \). In other words, there exists a positive constant \( M_1 \) such that for any \( f \in H^{s+\frac{1}{2}}(\Gamma) \)

\[ \left| \mathcal{K}f \right|_{s-\frac{1}{2}, \Gamma} \leq M_1 \left| f \right|_{s, \Gamma} \]

**Proof** By (20) and the definition of norm on \( H^{s}(\Gamma) \), for any \( f \in H^{s+\frac{1}{2}}(\Gamma) \) we have,

\[ \left| \mathcal{K}f \right|_{s-\frac{1}{2}, \Gamma} \leq \left| \sum_{n=-\infty}^{\infty} \left[ 1 + \left( \frac{n\pi}{\alpha} \right)^2 \right]^{s-\frac{1}{2}} \frac{\pi}{R\alpha} e_n Z_n u_n \right| \]

\[ \leq \left| \sum_{n=-\infty}^{\infty} \left[ 1 + \left( \frac{n\pi}{\alpha} \right)^2 \right]^{s-\frac{1}{2}} \frac{\pi}{R\alpha} e_n Z_n u_n \right| \]

Which completes the proof of Theorem 1.

**Theorem 2.** The bilinear form \( \tilde{D}(\bullet, \bullet) \) is symmetric and continuous on \( H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \), and \( H^{\frac{1}{2}}(\Gamma) \) coercive in the sense that exist two positive constants \( M_2 \) and \( M_3 \) such that for any \( u, v \in H^{\frac{1}{2}}(\Gamma) \)

\[ \tilde{D}(u, v) = \tilde{D}(v, u) \]

\[ \tilde{D}(u, v) \leq M_2 \cdot \| u \|_{\frac{1}{2}, \Gamma} \cdot \| v \|_{\frac{1}{2}, \Gamma} \]

\[ \tilde{D}(u, u) \geq M_3 \cdot \| u \|_{\frac{1}{2}, \Gamma} \]

**Proof** We have for any \( u, v \in H^{\frac{1}{2}}(\Gamma) \)

\[ \tilde{D}(u, v) = \mathcal{K}u \cdot v = \int_\Gamma \mathcal{K}u \cdot v dS \]

\[ = \int_0^\alpha \left( \frac{\pi}{\alpha} \sum_{n=-\infty}^{\infty} e_n Z_n u_n \cdot (e^{in\theta} + e^{-in\theta}) \right) \]

\[ \cdot \left( \sum_{n=-\infty}^{\infty} v_m (e^{im\theta} + e^{-im\theta}) \right) R d\theta \]

\[ = \frac{\pi}{\alpha} \int_0^\alpha \left( \sum_{n=-\infty}^{\infty} e_n Z_n u_n \cdot (e^{in\theta} + e^{-in\theta}) \right) \]

\[ \cdot \left( \sum_{n=-\infty}^{\infty} (2\alpha e_m Z_n u_n v_n) \right) \]

Hence, from this and Theorem 1 we have

\[ \tilde{D}(u, v) = 2\pi \sum_{n=-\infty}^{\infty} e_n Z_n u_n v_n = \tilde{D}(v, u) \]

\[ \tilde{D}(u, u) = \mathcal{K}u \cdot \| u \|_{\frac{1}{2}, \Gamma} \]

\[ \leq \| u \|_{\frac{1}{2}, \Gamma} \cdot \| v \|_{\frac{1}{2}, \Gamma} \]
\[ Q \cdot U = b \]

\[
\begin{bmatrix}
q_{00} & q_{01} & \cdots & q_{0,N-1} & q_{0,N} \\
q_{10} & q_{11} & \cdots & q_{1,N-1} & q_{1,N} \\
q_{20} & q_{21} & \cdots & q_{2,N-1} & q_{2,N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{N,0} & q_{N,1} & \cdots & q_{N,N-1} & q_{N,N} \\
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_N \\
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_N \\
\end{bmatrix}
\]  

(28)

where \( Q = (q_{ij})_{(N+1) \times (N+1)} \), \( q_{ij} = \hat{D} \left( L_i(\theta), L_j(\theta) \right) \), \( U = (U_0, U_1, \ldots, U_N)^T \), \( b = (b_0, b_1, \ldots, b_N)^T \), \( b_i = \hat{F}_n(L_i(\theta)) \).

Using (15), it is not difficult to get that \( q_{ij} \) is expressed as follows

\[ q_{ij} = q_{ji} = \frac{\pi}{\alpha} \sum_{n=0}^{\infty} \epsilon_n \cdot q_j(n) \cdot q_j(n), \]  

(29)

\( i, j = 0, 1, 2, ..., N \)

\[ q_k(n) = \int_0^\alpha L_k(\theta) \cos \frac{n\pi \theta}{\alpha} d\theta, \quad k = 0, 1, ..., N. \]  

(30)

where

\[ q_0(n) = \begin{cases} 
\frac{1}{2} h_1, & n = 0, \\
\frac{2\alpha^2}{n^2\pi^2h_1} \sin^2 \left( \frac{n\pi \theta_1}{2\alpha} \right), & n = 1, 2, \ldots 
\end{cases} \]  

(31)

\[ q_N(n) = \begin{cases} 
\frac{1}{2} h_N, & n = 0, \\
-\frac{2\alpha^2}{n^2\pi^2h_N} \sin \frac{n\pi(\theta_N + \theta_{N-1})}{2\alpha}, & n = 1, 2, \ldots \\
+\frac{n\pi(\theta_N - \theta_{N-1})}{2\alpha}, & n = 1, 2, \ldots 
\end{cases} \]  

(32)

\[ q_k(n) = \begin{cases} 
\frac{1}{2} (h_k + h_{k+1}), & n = 0, \\
-\frac{\alpha^2}{n^2\pi^2} \left( \frac{1}{h_k} \cos \frac{n\pi \theta_{k-1}}{\alpha} \right), & n = 1, 2, \ldots \\
+\frac{1}{h_k} \left( 1 + \frac{1}{h_{k+1}} \right) \cos \frac{n\pi \theta_{k+1}}{\alpha}, & n = 1, 2, \ldots 
\end{cases} \]  

(33)

\[ k = 1, 2, \ldots, N - 1. \]

Especially, we partition the boundary \( \Gamma \) into uniform subdivision, that is \( h = h_i = \frac{\alpha}{N}, \theta_i = i \cdot h = i \cdot \frac{\alpha}{N}, \) we now have

**FINITE ELEMENT DISCRETIZATION**

Now we consider the approximation of the problem (P). To the end, we discrete the circular arc \( \Gamma \) into a finite number of element domains and let \( S^h \) be the finite element subspace of space \( H^2(\Gamma) \) corresponding to the subdivision and spanned by some type of shape functions, that is \( S^h \subset H^2(\Gamma) \), then the discrete problem of variational problem 19:

\[(P_h) \text{ Find } u^h \in S^h \text{ such that } \]

\[ \hat{D}(u^h, v^h) = \hat{F}_n(u^h), \quad \forall v^h \in S^h \quad (25) \]

**1. Linear Element**

We now partition the boundary \( \Gamma \) into \( N \) parts, the points of division are \( \theta_i, i = 1, 2, \ldots, N - 1, \) i.e., \( 0 = \theta_0 < \theta_1 < \theta_2 \ldots < \theta_i < \theta_{i+1} \ldots < \theta_{N-1} < \theta_N = \alpha \), and set \( h_i = \theta_i - \theta_{i-1}, i = 1, 2, \ldots, N \). Let \( \{L_i(\theta)\} \) be the system of linear interpolation functions on \( \Gamma \). It is easy to know that \( \sum_{i=0}^{N} L_i(\theta) = 1 \) and \( S^h = \{L_i(\theta)\}_{i=0}^{N} \subset H^2(\Gamma) \).

Then

\[ u^h(R, \theta) = \sum_{i=0}^{N} U_i \cdot L_i(\theta) \quad (26) \]

Substituting (26) into (25), we easily obtain the system of linear algebraic equations of variational problem (25) as follows:
\[
q_0(n) = \begin{cases} \frac{\alpha}{2N}, & n = 0, \\ \frac{2n^2 \pi}{n^2 \pi^2} \sin^2 \left( \frac{n \pi}{2N} \right), & n = 1, 2, \ldots \end{cases} \tag{34}
\]

\[
q_n(n) = \begin{cases} \frac{\alpha}{2N}, & n = 0, \\ (-1)^{n+2} \frac{2n^2 \pi}{n^2 \pi^2} \sin^2 \left( \frac{n \pi}{2N} \right), & n = 1, 2, \ldots \end{cases} \tag{35}
\]

\[
q_k(n) = \begin{cases} \frac{\alpha}{N}, & n = 0, \\ -\frac{4N^2 \alpha}{n^2 \pi^2} \sin^2 \left( \frac{n \pi}{2N} \right) \cdot \cos \left( \frac{k}{N} \pi \right), & n = 1, 2, \ldots 
\end{cases} \tag{36}
\]

\[
q_{k-1}(n) = \begin{cases} \frac{(h_{2k} + h_{2k-1})^2 (2h_{2k} - h_{2k-1})}{n^2 \pi^2} & n = 0, \\ \frac{\alpha}{n^2 \pi} \frac{1}{h_{2k} h_{2k-1}} \left( \frac{h_{2k}}{h_{2k-1}} \right) \cos \left( \frac{n \pi \theta_{2k}}{\alpha} \right) & n = 1, 2, \ldots 
\end{cases} \tag{37}
\]

2. Quadratic Element

We now partition boundary \( \Gamma \) into \( 2N \) parts, the points of division are \( \theta_i, i = 1, 2, \ldots, 2N - 1 \), i.e., \( 0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_i < \theta_{i+1} < \ldots < \theta_{2N-1} < \theta_{2N} = \alpha \), and set \( \theta_j := \theta_i - \delta_0, i = 1, 2, \ldots, 2N \). Let \( \mathcal{Q}(\theta) \) be the system of quadratic interpolation functions on boundary \( \Gamma \), it is easy to know that \( \sum_{i=0}^{2N} \phi_i(\theta) = 1 \) and \( S^k = \{ \phi_i(\theta) \}_{i=0}^{2N} \subset H^2(\Gamma) \). Similar to the case of linear boundary element, we easily obtain the system of linear algebraic equation of problem (25), which is similar to the case of linear boundary element, Equation (27). However, \( Q = (q_{ij})_{(2N+1) \times (2N+1)} \), and \( q_{ij} \) are given by the followings:

\[
q_{ij} = q_{ji} = \frac{\pi}{\alpha \pi} \sum_{n=0}^{N} Z_n \cdot q_i(n) \cdot q_j(n),
\]

\[
i, j = 0, 1, 2, \ldots, 2N - 1, 2N
\]

\[
q_k(n) = \left\{ \phi_k(\theta) \cos \left( \frac{n \pi \theta_k}{\alpha} \right) d\theta \right\},
\]

\[
k = 1, 2, \ldots, 2N - 1, 2N
\]

CONVERGENCE AND PRIORI ERROR ESTIMATES

Let \( u_0 \) be the solution of natural integral equation (15), and \( u_h \) be the corresponding solution of natural boundary element, which is the solution of problem \( (P_h) \). The parameter \( h \) is stated as in section 3.

\[\| \Gamma_0^* \| := \sqrt{D(\Gamma_0^*)}. \] Let \( \Gamma \) be the circular arc of boundary of domain \( \Omega \) or \( \Omega^c \). In this section we will present the convergence and error estimates on boundary \( \Gamma \) for the numerical solution of the natural integral equation.

For the natural boundary element of elliptic boundary-value problems, under a united frame in Reference [10] has acquired some theorems with respect to the convergence and error estimates on boundary \( \Gamma \) for the numerical solution of natural integral equation. Since
...the inferences of these theorems do not depend on the concrete expression of natural integral operator \( \mathcal{K}_c \) so we can directly write out the following theorems.

**Theorem 3. (convergence)** The numerical solution \( u_0^h \) obtained by natural boundary element converges to the exact solution \( u_0 \) in terms of energy norm \( \| \cdot \|_\beta \) that is

\[
\lim_{h \to 0} \left\| u_0 - u_0^h \right\|_\beta = 0
\]

(43)

**Theorem 4. (priori error estimates)** If \( u_0 \in H^{r+1} (\Gamma) \), then the solution of natural integral equation has the following priori error estimates:

\[
\left\| u_0 - u_0^h \right\|_{L^2 (\Omega)} \leq C h^r \left\| u_0 \right\|_{r+1, \Gamma}, \quad \text{(L}^2 \text{ - norm)}
\]

(44)

\[
\left\| u_0 - u_0^h \right\|_{L^2 (\Gamma)} \leq C h^{r+1} \left\| u_0 \right\|_{r+1, \Gamma}. \quad \text{(L}^2 \text{ - norm)}
\]

(45)

\[
\left\| u_0 - u_0^h \right\|_{L^\infty (\Gamma)} \leq C h^{r+2} \left\| u_0 \right\|_{r+1, \Gamma}. \quad \text{(continuous norm)}
\]

(46)

where \( r (r \text{ is a positive integer number}) \) is the degree of piecewise interpolation polynomial, \( C \) a positive constant independent of \( u_0, h, \alpha \) and \( R \). (45) and (46) hold only for \( u_0 \) satisfying \( \int_{\alpha}^{\beta} [u_\alpha (R, \theta) - u_\beta (R, \theta)] d\theta = 0 \), while \( \beta = 0 \).

**NUMERICAL EXPERIMENTS**

In this section we only find the approximation \( u_n^\alpha (R, \theta) \) of \( u_0 (R, \theta) \) by using the natural boundary element method stated as the above. In fact, once the approximation \( u_n^\alpha (R, \theta) \) of \( u_0 (R, \theta) \) is obtained, the approximation \( u_n (r, \theta) \) of the exact solution \( u (r, \theta) \) in \( \Omega \) or \( \Omega^c \) can be acquired directly by Poisson integral formulæ. We here omit the part of computation. Since the elements of stiffness matrix \( Q \) are given by some infinite series, we substitute \( \sum_{n=1}^{M} \) into \( \sum_{n=1}^{\infty} \) in infinite series in practice computing the elements of stiffness matrix \( Q \) (generally we take \( M \leq 100 \)). The system of linear algebraic equations obtained by finite element discretization can be solved by Gauss-Seidel iteration method, and the number of iteration is denoted by \( M_1 \).

Denote the maxmal error of \( L^2 \)-norm on boundary by \( E_{\text{max}} \). To sure the problem (1)-(5) has a unique solution, some conditions must be added for \( \beta = 0 \). \( U_N (\mathcal{N}, \alpha) \) is taken to be zero while linear boundary element is used, and \( U_N \) to be zero while quadratic boundary element is used.

**Example 1.** Solving the problem in exterior domain \( \Omega^c \), where \( R = 1, \alpha = \frac{3\pi}{4} \), and function \( g_n (R, \theta) \) is as follows

\[
g_n (R, \theta) = \begin{cases} 
K_0 (R) \cdot \cos \frac{4\theta}{5}, & \beta = -1; \\
\cos \frac{4\theta}{5}, & \beta = 0; \\
H_4 (R) \cdot \cos \frac{4\theta}{5}, & \beta = 1.
\end{cases}
\]

This problem has the exact solution:

\[
u (r, \theta) = \begin{cases} 
- K_0 (r) \cdot \cos \frac{4\theta}{5}, & \beta = -1; \\
\frac{5}{4} r^\frac{3}{5} \cos \frac{4\theta}{5}, & \beta = 0; \\
- H_4 (r) \cdot \cos \frac{4\theta}{5}, & \beta = 1.
\end{cases}
\]

where \( r > R \) and \( 0 < \theta < \alpha \). Numerical results are shown by Tables 1 and 2, respectively.

**Example 2.** Solving the problems in interior domain \( \Omega \), where \( R = 1, \alpha = \frac{3\pi}{4} \), and function \( g_n (R, \theta) \) is as follows

\[
g_n (R, \theta) = \begin{cases} 
J_0 (R) \cdot \cos \frac{4\theta}{5}, & \beta = -1; \\
\cos \frac{4\theta}{5}, & \beta = 0; \\
J_4 (R) \cdot \cos \frac{4\theta}{5}, & \beta = 1.
\end{cases}
\]

Now the exact solution of the problem is

\[
u (r, \theta) = \begin{cases} 
J_0 (r) \cdot \cos \frac{4\theta}{5}, & \beta = -1; \\
\frac{5}{4} r^\frac{3}{5} \cos \frac{4\theta}{5}, & \beta = 0; \\
J_4 (r) \cdot \cos \frac{4\theta}{5}, & \beta = 1.
\end{cases}
\]

where \( r < R \) and \( 0 < \theta < \alpha \). Numerical results are shown by Tables 3 and 4, respectively.

**Table 1. Linear boundary element (piecewise linear element),** \( R = 1, \alpha = \frac{3\pi}{4} \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M )</th>
<th>( M_1 )</th>
<th>( E_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>20</td>
<td>20</td>
<td>1.55E-2 2.11E-2 1.68E-2</td>
</tr>
<tr>
<td>16</td>
<td>40</td>
<td>60</td>
<td>3.96E-3 5.56E-3 4.32E-3</td>
</tr>
<tr>
<td>32</td>
<td>80</td>
<td>100</td>
<td>1.03E-3 1.43E-3 1.12E-3</td>
</tr>
</tbody>
</table>

**Table 2. Quadratic boundary element (piecewise quadratic element),** \( R = 1, \alpha = \frac{3\pi}{4} \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M )</th>
<th>( M_1 )</th>
<th>( E_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>20</td>
<td>20</td>
<td>1.24E-2 2.53E-2 1.41E-2</td>
</tr>
<tr>
<td>16</td>
<td>40</td>
<td>60</td>
<td>1.64E-3 2.41E-3 1.84E-3</td>
</tr>
<tr>
<td>32</td>
<td>80</td>
<td>100</td>
<td>2.10E-4 3.07E-4 2.34E-4</td>
</tr>
</tbody>
</table>
Table 3. Linear boundary element (piecewise linear element), $R = 1, \alpha = \frac{5\pi}{4}$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>$M_1$</th>
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<td>1.13E-3</td>
</tr>
</tbody>
</table>

Table 4. Quadratic boundary element (piecewise quadratic element), $R = 1, \alpha = \frac{5\pi}{4}$

<table>
<thead>
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<th>$M$</th>
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</tr>
<tr>
<td>32</td>
<td>80</td>
<td>100</td>
<td>2.13E-4</td>
</tr>
</tbody>
</table>

**SOME APPLICATIONS**

Let $\Omega'$ be an unbounded domain whose boundary is composed of two sides $\Gamma_0$ and $\Gamma_\alpha$ of a concave angle $0 < \alpha < 2\pi$ and a simple curve $\Gamma$, i.e. $\Gamma$ is defined by any single-valued continuous function of $\theta$, $\theta \in (0, \alpha)$, $\Gamma = \{(r, \theta) \mid r = r(\theta), \theta \in (0, \alpha)\}$ and $r = r(\theta)$ is a single valued function. Considering the following value problem:

\[
\begin{aligned}
\Delta u + \beta u &= 0 \quad \text{in} \quad \Omega^c \\
\frac{\partial u}{\partial n}(r, 0) &= 0, \quad \text{on} \quad \Gamma_0 \\
\frac{\partial u}{\partial n}(r, \alpha) &= 0, \quad \text{on} \quad \Gamma_\alpha \\
\frac{\partial u}{\partial n}(r, \theta) &= g_n(r, \theta), \quad \text{on} \quad \Gamma_\alpha \\
\end{aligned}
\]

Conditions at infinity

In $\Omega'$ we draw a circular arc $R = \{(R, \theta) \mid 0 \leq \theta \leq \alpha\}$, it divides $\Omega'$ into $\Omega_1$ and $\Omega_2$. $\Omega_2$ is an infinite sector. It is not difficult to get that (47) is equivalent to

\[
\begin{aligned}
\text{Find } u \in H^1(\Omega_1) \text{ such that for any } v \in H^1(\Omega_1) \\
D_1(u, v) + \tilde{D}_2(u, v) &= <g_n, v> \quad \Gamma
\end{aligned}
\]

where $D_1(u, v) = \int_{\Omega_1} (\nabla u \cdot \nabla v - \beta u v) \, dx, \tilde{D}_2(u, v) = \int_{\Gamma} v \cdot (\partial \mathbf{u}/\partial n) \, dS$.

\[
Ku(R, \theta) = \frac{\pi}{R \alpha} \sum_{n=0}^{\infty} e_n Z_n \int_{0}^{\alpha} u(R, \theta) \cos \frac{n\pi\theta}{\alpha} \cos \frac{n\pi\theta}{\alpha} \, d\theta
\]

Now we use the finite element method in $\Omega_1$. Set $V = (U_0, U_1, ..., U_N)^T$ is a set of values at nodes on $\Gamma_R$, $U = (U_{N+1}, U_{N+2}, ..., U_{N+M})^T$ is set of values at nodes on $\Gamma$ and at interior nodes. $\{L_i(x, y)\}_{i=0}^{M+N} \subseteq H^1(\Omega_1)$ are corresponding basis function, for example, piecewise linear, then restriction of $\Gamma_R$ are approximately piecewise linear on $\Gamma_R$. Let $u = \sum_{i=0}^{N} U_i \cdot L_i(x, y)$, we obtain

\[
\begin{aligned}
\sum_{j=0}^{N} D_1(L_i, L_j) U_j + \sum_{j=0}^{N} \tilde{D}_2(L_i, L_j) U_j \\
\end{aligned}
\]

\[
\begin{aligned}
K + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} V = \begin{pmatrix} b_0 \\ b \end{pmatrix}
\end{aligned}
\]

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CONCLUDING REMARKS

We have derived a sequence of natural integral equations for solving elliptic problems with concave angle domains, including problems in interior domain and exterior domain in plane. A finite element formulation is presented in computing natural integral equations. Error estimates for finite element approximation are given, which depend on the parameter $h$. From our numerical results, we can make several concluding remarks:

1. It takes much time to obtain the numerical integration by using classical boundary element method, especially to deal with singular integral. However, for natural boundary element method we have seen that the explicit expressions of these elements of boundary element stiffness matrix are given (see Eqs. (29)-(33) or (37)-(42)). And they have some distinctive properties. It is easy to be implemented on calculation and storage comparing with classical boundary element methods.

2. To solve natural integral equation is very simple and easy in programming. At the same time, natural integral equation can be used as the artificial boundary condition in practice, and one can get better accuracy for solving problems in unbounded domains by standard finite element methods. Thus we recommend engineers to use the method.

3. Some domain decomposition methods based on natural boundary reduction have been used to solve some elliptic problems with unbounded domains or...
We shall report on progress in some of these directions in a future publication.

ACKNOWLEDGMENT

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REFERENCES


四角型区域椭圆边界值问题的自然边界元法

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摘 要

本文研究四角型區域椭圆邊值問題的自然邊界

歸化及其邊界元法。給出了自然總分方程及相應的

Poisson總公式與近似解的收敛性及誤差估計，詳

細地討論了自然總分方程的有限元数值方法，我們給

出數值試驗以說明方法的有效性。作爲應用，我們

也研究了無限扇形領域的有限元與自然邊界元耦合

法。

關鍵詞：自然邊界歸化；四角區域；數值方法；耦

合。