

# THE LEAST SQUARES TREFFTZ METHOD AND THE METHOD OF EXTERNAL SOURCE FOR THE EIGENFREQUENCIES OF WAVEGUIDES

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Key words: least squares Trefftz method, eigenfrequencies problem, Helmholtz equation, boundary-type meshless method.

## ABSTRACT

In this study, the least squares Trefftz method (LSTM) is adopted for analyzing the eigenfrequencies problems governed by homogeneous Helmholtz equations. The Trefftz method, one kind of boundary-type meshless collocation methods, does not need mesh generation and numerical quadrature. Since the system of linear algebraic equations obtained by Trefftz method is highly ill-conditioned, the least squares method is adopted to stabilize the numerical scheme in this study. In the eigenproblem, the response amplitudes from an external source are used to determine the resonant frequencies. By adding an external source, the homogeneous boundary condition becomes inhomogeneous. Then we can employ the LSTM to easily solve this problem. In this paper, the LSTM and the method of external source are used to solve this eigenfrequencies problems governed by Helmholtz equations. Several numerical examples are provided to verify the accuracy and the simplicity of the proposed numerical scheme.

## I. INTRODUCTION

Waveguide is a thin-tube device and is used to transfer and guide electromagnetic waves. Since it can transfer electromagnetic wave, it is very useful in many electronic applications and is also an important device in optics. The determination of eigenfrequencies of the waveguide is important when the electromagnetic waves of specific frequency have to propagate in the designed direction. In order to resolve the eigenfrequencies problems, many researchers developed numerical methods for acquiring the eigenfrequencies of waveguides in the past, such as Chen et al. [3], Dong et al. [7], Fan et al. [10], Kuttler [16], Lin et al. [18], Reutskiy [22], Tsai et al. [27], Young et al. [28] etc.

With the rapid developments of computer equipments, there are many numerical schemes, which are proposed to solve engineering problems and can be classified as the mesh-dependent and the meshless methods. The meshless numerical methods do not need mesh generation and numerical quadrature, so they will cost less computational resources, such as the method of fundamental solutions (MFS) [5, 9, 14, 26, 29] the radial basis functions collocation method (RBFCM) [6, 8, 15, 21, 30], the meshless Galerkin method [2], the Trefftz method [1, 4, 11, 12, 13, 17, 19, 20] etc. Jiang et al. [15] used the RBFCM, one of the popular meshless methods, to analyze the eigenproblems of elliptic waveguides. The boundary nodes and interior nodes are all required during the computation; hence the requirement of huge number of nodes in the simulation will limit the applications of the RBFCM. Young et al. [28] adopted the MFS with the singular value decomposition (SVD) technique to solve the eigenproblems of waveguides. The homogeneous partial differential equations and homogeneous boundary conditions are both important in eigenproblems. Reutskiy [22, 23, 24, 25] recently proposed a novel numerical scheme by utilizing the method of external source (MES). The time-consuming SVD or direct determinant search method (DDSM) for dealing with the eigenfrequencies problems is no longer required in the MES. Following the lead of Reutskiy, Fan et al. [10] used the MFS and the MES to determine the eigenfrequencies of waveguides.

The boundary-type meshless methods only need the boundary information rather than the mesh of the computational domain, so these methods are simple and easy to be implemented. The Trefftz method is one kind of boundary-type meshless methods and is more suitable for the eigenproblems. The numerical solutions of the Trefftz method can be expressed as the linear combinations of T-complete functions such that we only need to require the satisfactions of the boundary conditions on the collocated boundary points. Since the governing equations in the eigenfrequencies problems is the Helmholtz equation, the characteristic length in modified collocation Trefftz method [19,20] is incapable of reducing the ill-conditioned problem. So, we used the least squares method to ease the ill-conditioned problem in the Trefftz method.

In this study, we used the least squares method and the Trefftz method to form and resolve the system of linear algebraic equations. This proposed method, marked here as the LSTM, is used in this study to analyze the eigenfrequencies for different waveguides. Since the Trefftz method with external source can transfer the eigenproblem from a homogeneous problem into a series of inhomogeneous problems, the eigenfrequencies can be determined by solving a series of direct problems. We will describe the governing equations and the LSTM in the following sections. Then the numerical results and comparisons of square, elliptic, concentric annular and eccentric annular waveguides are provided in the section of numerical results to validate the accuracy of the proposed method.

## II. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

### 1. Governing Equation

Maxwell's equations system is an important system of partial differential equations used to describe the electromagnetic phenomena. The Maxwell's equations are depicted as follows:

$$\nabla \cdot \bar{B} = 0, \quad (1)$$

$$\nabla \cdot \bar{D} = \rho, \quad (2)$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \quad (3)$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}, \quad (4)$$

where  $\bar{B}$  and  $\bar{D}$  are magnetic flux density and electric flux density.  $\bar{E}$  and  $\bar{H}$  denote the electric field intensity and magnetic field intensity.  $\rho$  is the electric charge density and  $\bar{J}$  is the electric current density. By following some assumptions and mathematical derivations [10, 28], the governing equations of electromagnetic wave in frequency domain can be derived and be shown as:

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0, \quad (5)$$

$$\nabla^2 \bar{H} + k^2 \bar{H} = 0, \quad (6)$$

where  $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$  is the wavenumber.  $\omega$  is the angular frequency.  $c$  is the wave velocity.  $\lambda$  is the wavelength. Equations (5) and (6) are governing equations for electric field and magnetic field in frequency domain.

### 2. Boundary Conditions

In this waveguide problem, the governing equation is the homogeneous Helmholtz equation,

$$(\nabla^2 + k^2)u(x, y) = 0, \quad (x, y) \in \Omega. \quad (7)$$

The electromagnetic wave can be divided into two kinds of basic waves. When  $u(x, y) = E_z$ , it describes the transverse magnetic (TM) wave.  $u(x, y) = H_z$  represents the transverse electric (TE) wave.

The boundary condition in the TM wave is the homogeneous Dirichlet boundary condition,

$$E_z = 0. \quad (8)$$

For the TE wave, the homogeneous Neumann boundary condition is imposed

$$\frac{\partial H_z}{\partial n} = 0. \quad (9)$$

The eigenfrequencies problem of waveguide is to determine the resonant wavenumbers and the numerical solutions for the homogeneous governing equation and homogeneous boundary condition. The schematic diagram for the eigenfrequencies problem is demonstrated in Fig. 1.

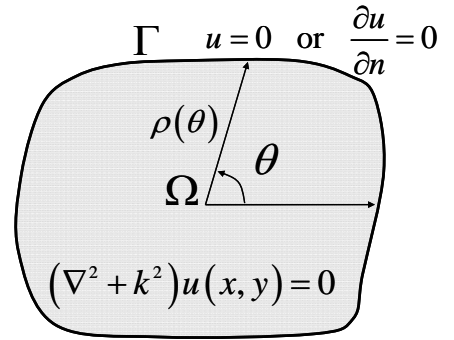


Fig. 1. The schematic diagram for the two-dimensional eigenproblem.

## III. NUMERICAL METHOD

### 1. Trefftz Method and Method of External Source

In this study, the Trefftz method is used to solve the Helmholtz equation. The T-complete functions for two-dimensional Helmholtz equation for the simple-connected and the doubly-connected domains are depicted in Eq. (10) and Eq. (11),

$$\{J_0(kr), J_j(kr)\cos(j\theta), J_j(kr)\sin(j\theta), j=1, 2, 3, \dots\}, \quad (10)$$

$$\left\{ \begin{array}{l} J_0(kr), J_j(kr)\cos(j\theta), J_j(kr)\sin(j\theta), \\ Y_0(kr), Y_j(kr)\cos(j\theta), Y_j(kr)\sin(j\theta), j=1, 2, 3, \dots \end{array} \right\}, \quad (11)$$

where  $J_j(\cdot)$  and  $Y_j(\cdot)$  are the Bessel functions of the first kind and the second kind.

For problems in simply- and doubly-connected domains, the numerical solutions can be expressed by the linear combinations of the T-complete functions listed in Eq. (10) and Eq. (11). The corresponding outer boundary of the computational domain,  $\Omega$ , in the polar coordinates is given by  $\Gamma_0 = \{(r, \theta) | r = \rho(\theta), 0 \leq \theta \leq 2\pi\}$  and the corresponding inner boundary is given by  $\Gamma_1 = \{(r, \theta) | r = \eta(\theta), 0 \leq \theta \leq 2\pi\}$ . The numerical solutions of the Helmholtz equation for simply- and doubly-connected domains in the Trefftz method can be expressed by the linear combinations of the T-complete functions.

$$u(x, y) = a_0 J_0(kr) + \sum_{j=1}^N a_j J_j(kr) \cos(j\theta) + b_j J_j(kr) \sin(j\theta), \quad (12)$$

$$u(x, y) = a_0 J_0(kr) + c_0 Y_0(kr) + \sum_{j=1}^N a_j J_j(kr) \cos(j\theta) + b_j J_j(kr) \sin(j\theta) + c_j Y_j(kr) \cos(j\theta) + d_j Y_j(kr) \sin(j\theta), \quad (13)$$

where  $\{a_j\}_{j=0}^N$ ,  $\{b_j\}_{j=1}^N$ ,  $\{c_j\}_{j=0}^N$  and  $\{d_j\}_{j=1}^N$  are the unknown coefficients that will be retrieved by enforcing the satisfactions of boundary conditions on the boundary collocation points. In Eq. (12) and Eq. (13), terms up to the  $N$ -th order are used to replace the infinite series in the original expressions. Once the unknown coefficients are obtained, the numerical solutions and its derivatives at any positions inside the computational domain can be found from Eq. (12) and Eq. (13).

If we directly solve this eigenfrequencies problem by Trefftz method, the boundary condition is homogeneous along the whole boundary. So, it is non-trivial to solve the unknown coefficients  $\{a_j\}_{j=0}^N$ ,  $\{b_j\}_{j=1}^N$ ,  $\{c_j\}_{j=0}^N$  and  $\{d_j\}_{j=1}^N$  by direct collocation. In a study by Reutskiy [22], she proposed the MES to transfer the eigenproblem. By adding an external source, the eigenfrequencies problem will be converted to a homogeneous Helmholtz equation with inhomogeneous boundary condition. The external source can be located at any place other than the computational domain [10]. The position of the external source is denoted by  $\bar{X}_{ext} = (x_{ext}, y_{ext})$ . The inhomogeneous governing equation is in the following form

$$(\nabla^2 + k^2)u(\bar{x}) = \delta(\bar{x} - \bar{X}_{ext}), \quad (14)$$

where  $\delta$  is the Dirac delta function. Thus, the numerical solution can be divided into the homogeneous solution and the particular solution:

$$u(\bar{x}) = u_h(\bar{x}) + u_p(\bar{x}), \quad (15)$$

where  $u_h(\bar{x})$  is the homogeneous solution and  $u_p(\bar{x})$  is the particular solution. The particular solution is also the fundamental solution of the Helmholtz equation and can be obtained by using the Fourier transform theory:

$$u_p(\bar{x}) = \frac{i}{4} H_0^{(2)}\left(k|\bar{x} - \bar{X}_{ext}|\right), \quad (16)$$

where  $H_0^{(2)}(\cdot)$  is the Hankel function of the second kind of zero order.

Then, we use the Trefftz method to solve the homogeneous solution with inhomogeneous boundary condition. The inhomogeneous boundary condition is derived from the particular solution. The particular solution satisfies the inhomogeneous equation in Eq. (14) without boundary condition. Now the eigenproblem is converted to a Helmholtz equation with inhomogeneous boundary conditions and it can be shown as:

$$\nabla^2 u_h(\bar{x}) + k^2 u_h(\bar{x}) = 0, \quad (17)$$

$$G_{B.C.} [u_h(\bar{x})] = -G_{B.C.} [u_p(\bar{x})], \quad (18)$$

where  $G_{B.C.}[\cdot]$  is the partial differential operator for boundary condition. So, it is easy to solve the solution of the above system by using the LSTM instead of the original eigenfrequencies problem.

For a study range of wavenumber, we resolve the Helmholtz problems from Eq. (17) and Eq. (18) by using different wavenumbers. Then, the resonant responses of the numerical solutions by adopting different wavenumbers are recorded. The resonant responses is calculated by the following equation:

$$F(k) = \sqrt{\frac{1}{N_r} \sum_{j=1}^{N_r} |u_h(\bar{x}_j)|^2}, \quad (19)$$

$$F_d(k) = \frac{F(k)}{F(k_0)}, \quad (20)$$

where  $F_d(k)$  is a dimensionless value.  $k_0$  is a reference wavenumber which is set as unit in our study.  $N_r$  is the number of measurement points randomly distributed inside the domain. When the peak appears in the resonant curve, the eigenfrequency can be obtained. Following the same procedure, a series of eigenfrequencies can be acquired for a waveguide.

## 2. Least Squares Method

The least squares method is a standard approach to approximate solution of over-determined or under-determined systems. In the beginning of simulation,  $M$  boundary nodes will be distributed along the whole boundary, so a system of  $M$  linear algebraic equations will be formed by enforcing the satisfactions of boundary conditions ( $\mathbf{Ax} = \mathbf{B}$ ). On the other hand, there are  $2N + 1$  unknowns in the solution expression for simply-connected domain. In all of the numerical tests, the number of equations is greater than the number of unknowns, which will form an over-determined system.

For the linear algebraic equations that form is  $\mathbf{Ax} = \mathbf{B}$ , the residual error is defined as

$$\mathbf{R} = \mathbf{Ax} - \mathbf{b}, \quad (21)$$

where  $\mathbf{R}$  is the residual error matrix. The sum of the squared residuals is defined as

$$\mathbf{S} = \mathbf{R}^T \mathbf{R}, \quad (22)$$

where  $\mathbf{S}$  is the sum of the squared residuals. The minimum of  $\mathbf{S}$  is approximated by taking gradient

$$\min(\mathbf{S}) \cong \frac{\partial \mathbf{S}}{\partial \mathbf{x}} = 0. \quad (23)$$

Then, Eq. (21) and Eq. (22) are substituted into Eq. (23),

$$\min(\mathbf{S}) \cong \mathbf{A}^T \mathbf{Ax} - \mathbf{A}^T \mathbf{b} = 0. \quad (24)$$

Finally, the system can be rewritten as

$$\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}. \quad (25)$$

To solve Eq. (25) by any solvers for linear system can obtain the unknowns of the original system. We will use the least squares method to solve the matrix system from the Trefftz method. To use the least squares method will evidently reduce the ill-conditioned problem and stabilize the numerical scheme, which will be experimentally shown in the next section.

#### IV. NUMERICAL RESULTS AND COMPARISONS

The eigenfrequencies problem governed by the two-dimensional Helmholtz equation will be resolved by the proposed algorithm, the LSTM and the MES. In this paper, we will investigate the square, elliptic, concentric annular and eccentric annular waveguides shown in Fig. 2 to verify the accuracy and simplicity of the proposed numerical method. For clarity, the following abbreviations are used in these examples:  $M$  denotes the number of boundary nodes along  $\Gamma$ ,  $N$  denotes the order of Trefftz method,  $k$  is the wavenumber,

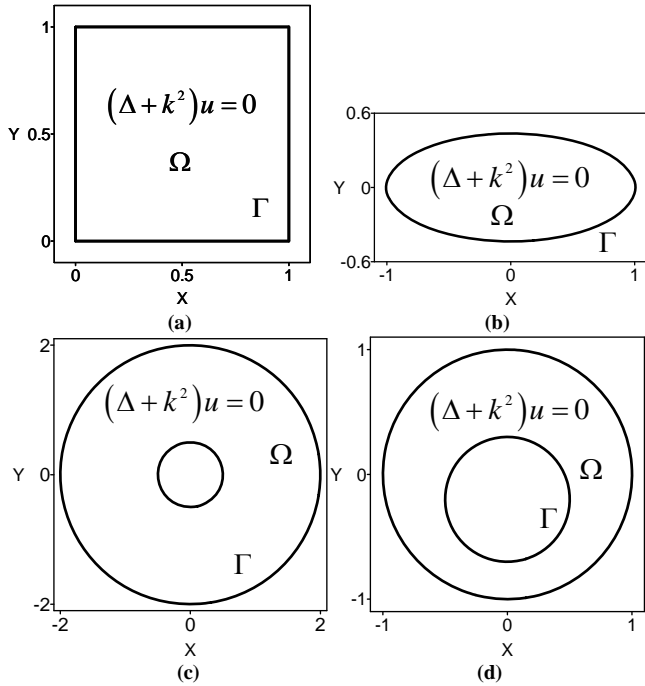


Fig. 2. The computational domain and corresponding boundary for (a) Example 1, (b) Example 2, (c) Example 3 and (d) Example 4.

##### 1. Example 1

In the first example, the square waveguide is the typical shape of eigenproblem and the corresponding resonance curve is demonstrated in Fig. 3. The parameters are set to be  $M = 73$ ,  $N=30$  and the external source is located at  $(10,10)$ .

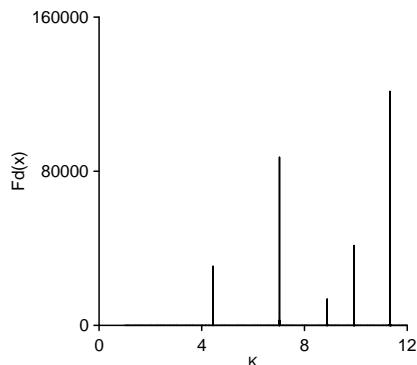


Fig. 3. Resonance curve for square waveguide.

We can easily find that there are five peaks appeared in the range from zero to twelve. In other words, there are five eigenfrequencies in this studying range. In Table 1, the numerical solutions are compared well with analytical solution and other numerical results obtained by the generalized differential quadrature (GDQ) method [7] and the MFS [10].

In Fig. 4(a) - Fig. 4(d), the former four eigenmodes for the TM wave of the square waveguide are shown respectively and they are very similar to the analytical solutions. Therefore, the ability of using the LSTM to acquire the eigenfrequencies of square waveguide is verified and the numerical solutions are very stable and accurate.

Table 1. Comparison of the present solutions with analytical and other numerical results in example 1.

	Analytical Solution	MFS-ES ( $M = 24$ ) [10]	MFS-ES ( $M = 32$ ) [10]	GDQ method ( $M = 324$ ) [7]	LSTM ( $M = 73$ )
1	4.4429	4.4429	4.4429	4.4429	4.4429
2	7.0248	7.0248	7.0248	7.0248	7.0248
3	8.8858	8.8857	8.8858	8.8857	8.8858
4	9.9346	9.9349	9.9346	9.9469	9.9346
5	11.3237	11.3267	11.3237	11.3448	11.3237

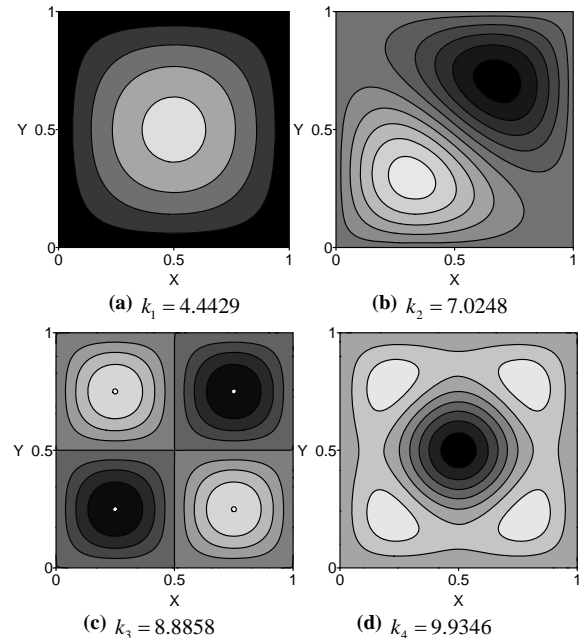
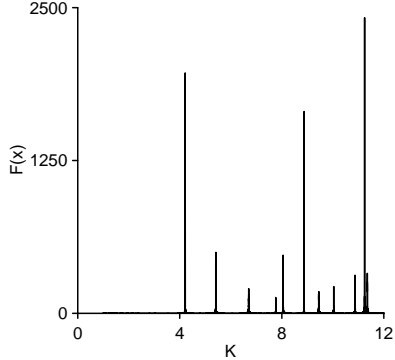


Fig. 4. The former four eigenmodes for square waveguide.

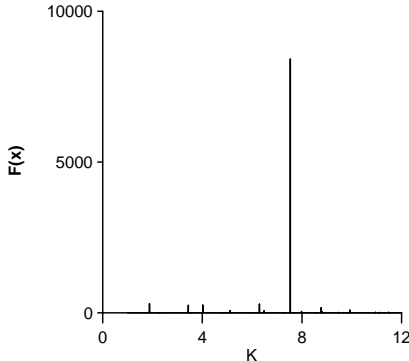
##### 2. Example 2

In the second example, we solved the eigenfrequencies problem of an elliptic waveguide which is defined by the parametric equation,

$\Gamma = \{(x, y) | x = a \cos \theta, y = b \sin \theta, 0 \leq \theta \leq 2\pi\}$ . The following parameters are used in example 2: eccentricity  $e = 0.9$ , major axis  $a = 1$ , minor axis  $b = \sqrt{1 - e^2}$ ,  $M = 120$ ,  $N = 25$ ,  $X_{ext} = (10, 10)$ .



(a) TM wave



(b) TE wave

Fig. 5. Resonance curves for elliptic waveguide for (a) TM wave and (b) TE wave.

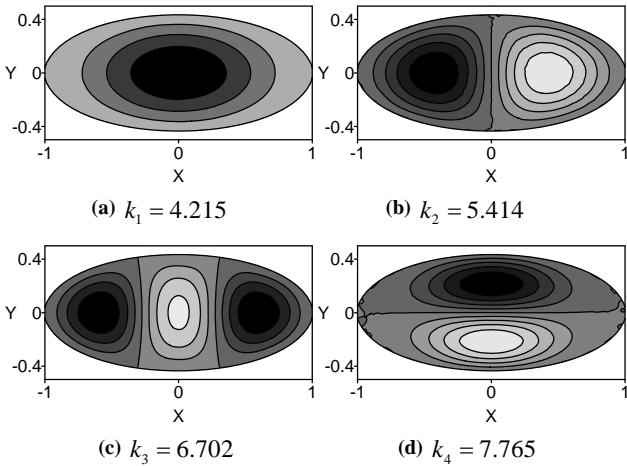


Fig. 6. The former four eigenmodes for elliptic waveguide (TM wave).

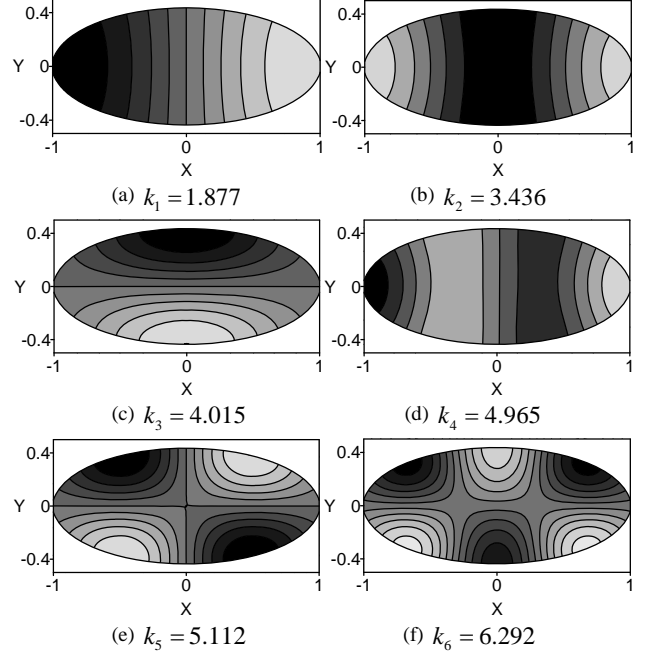


Fig. 7. The former six eigenmodes for elliptic waveguide (TE wave).

In Fig. 5(a), the resonance curve of the TM wave evidently shows the former eleven peaks in the range from zero to twelve. In Fig. 5(b), we can find many peaks in the resonance curve of the TE wave. The eigenfrequencies of the TM wave and the TE wave are compared well with other numerical solutions. The former four eigenmodes of the TM wave are shown in Fig. 6 and the former six eigenmodes of the TE wave are depicted in Fig. 7. The solutions of the eigenfrequencies are solved very well and accurately.

### 3. Example 3

In the third example, a concentric annular waveguide is considered. The radii of the outer and inner boundaries of the concentric annular waveguide are 2 and 0.5 respectively. The centers of the outer and inner boundaries are all  $(0, 0)$ . The following parameters are used in example 3:  $M = 60$ ,  $N = 10$ ,  $X_{ext} = (10, 10)$ .

The corresponding resonance curve for TM wave is demonstrated in Fig. 8. It is obvious that there are many peaks in the curve and the former four eigenmodes are displayed in Fig. 9(a) to Fig. 9(d), respectively. Table 2 lists the former five eigenfrequencies which are compared very well with analytical solution and other numerical solutions obtained by finite element method (FEM) [3], boundary element method (BEM) [3], and MFS-DDSM [27].

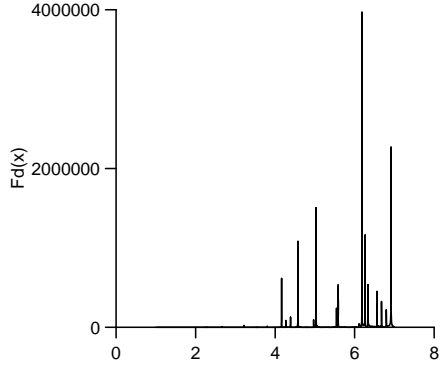


Fig. 8. Resonance curves for concentric annular waveguide. (TM wave).

Table 2. Comparison of the former five eigenfrequencies for concentric annular waveguide in example 3.

	Analytical Solution	FEM [3]	BEM [3]	MFS-DDSM [27]	LSTM ( $M = 60$ )
1	2.05	2.03	2.06	2.05	2.05
2	2.23	2.20	2.23	2.22	2.22
3	2.66	2.62	2.67	2.66	2.66
4	3.21	3.15	3.22	3.21	3.21
5	3.80	3.71	3.81	3.80	3.80

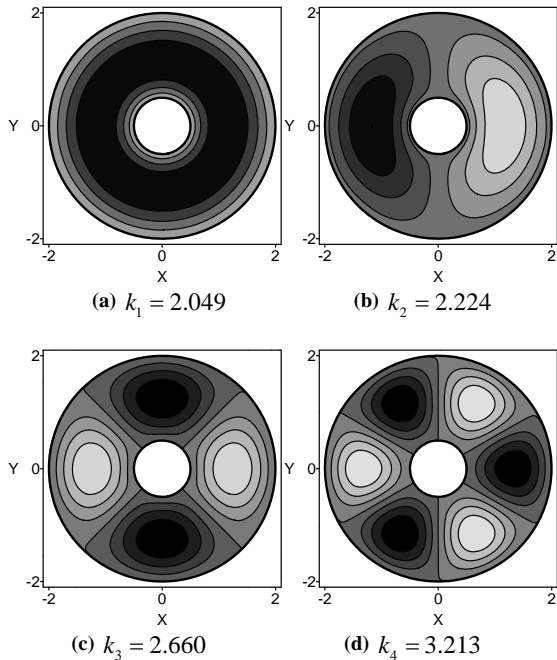


Fig. 9. The former four eigenmodes for concentric annular waveguide.

#### 4. Example 4

For the fourth example, we solved the eigenfrequencies problem for an eccentric annular waveguide which is a doubly-connected domain. The shape of the waveguide is the same as that in Ref. [16,18]. The following parameters are used in example 4:  $M = 64$ ,  $N = 10$ ,  $X_{ext} = (10,10)$ .

In Fig. 10, the resonance curve of the TM wave shows the former four peaks in the range from zero to eight. The former four eigenmodes in the TM wave are demonstrated in Fig. 11(a) to Fig. 11(d). The solutions of the eigenfrequencies are obtained stably and accurately. In Table 3 the comparisons of the eccentric annular waveguide with other researches, such as Kuttler [16], Lin et al. [18] and Fan et al. [10], are tabulated. In this test, the number of collocation points is less than 100 points and it still can quickly achieve good results.

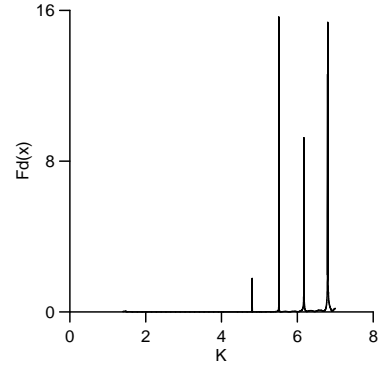


Fig. 10. Resonance curves for eccentric annular waveguide. (TM wave)

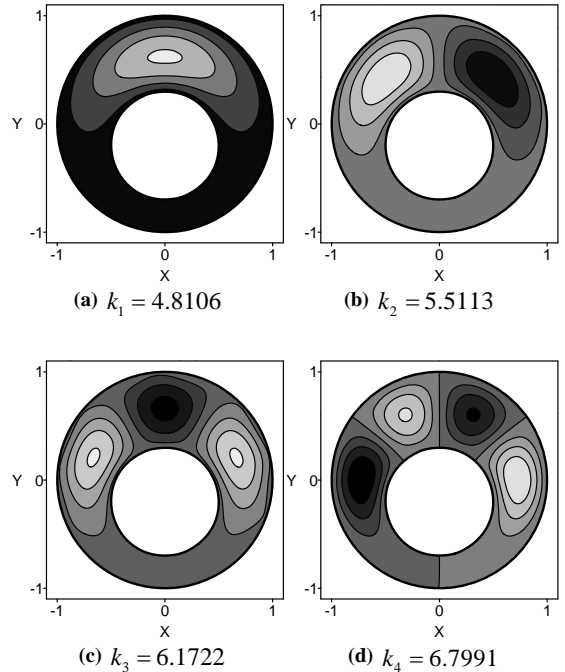


Fig. 11. The former four eigenmodes for eccentric annular waveguide.

**Table 3. Comparison of the former five eigenfrequencies for eccentric annular waveguide in example 4.**

	Lin et al. [18]	Kuttler [16]	MFS-MES ( $M=100$ ) [10]	MFS-MES ( $M=140$ ) [10]	LSTM ( $M=64$ )
1	4.8129	4.8119	4.8106	4.8106	4.8106
2	5.5252	5.5125	5.5114	5.5114	5.5113
3	6.2099	6.1735	6.1724	6.1724	6.1722
4	6.8375	6.8002	6.7991	6.7991	6.7991
5	7.4619	7.3957	7.3945	7.3945	7.3942

## V. CONCLUSIONS

In this paper, we used the combination of the meshless numerical method and the least squares method to acquire the eigenfrequencies in four different waveguides. The LSTM and the MES are used to solve this eigenfrequencies problems governed by two-dimensional Helmholtz equation. By adding an external source, the homogeneous boundary condition becomes inhomogeneous and we can simply employ the meshless Trefftz method to solve this system.

There are four examples: square, elliptic, concentric annular and eccentric annular waveguides. The numerical results are provided to validate the simplicity of the proposed LSTM. The resonant eigenfrequencies can be obtained from response figures. In comparing with other numerical results, the acquired eigenfrequencies are highly accurate and the numerical scheme is very stable. The numerical results for simply-connected domain and doubly-connected domain are all extremely accurate by using very few nodes. Finally, it is numerically verified that the proposed method is very stable and simple for solving the eigenfrequencies of waveguides.

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